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A strong complete semantics for Łukasiewicz propositional logic

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I use logic all the time in mathematics, and it seems to yield "correct" results, but in mathematics "correct" by and large means "logical", so I'm back where I started. I can't defend logic because I can't remove my glasses.

Richard J. Trudeau

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Introduction

The notion of *completeness* is one of the most important in mathematical logic since it links semantics with syntax. For instance in the classical propositional logic the notion of *tautology* can be given in two different ways. On the one hand we have the set of semantic tautologies which are those formulas τ such that, for each valuation v into the Boolean algebra {0,1}, $v(\tau) = 1$, in symbols $\models \tau$. On the other hand, we have the set of syntactic tautologies which are those formulas which can be derived from axioms by substitution and modus pones, in symbols τ is tautology if $\vdash \tau$. Thanks to the *completeness theorem* we know that these two approaches give the same set of formulas. With the birth of non-classical logic one wondered if the completeness theorem was still valid. In this thesis the attention is focused on the completeness in Łukasiewicz propositional logic $Ł_\infty$ that is a non-classical logic introduced by Jan Łukasiewicz and Alfred Tarski in 1930. As in classical propositional logic, in $Ł_\infty$ we can define the two sets of semantic and syntactic tautologies. In 1958 Alan Rose and J. Barkley Rosser give a proof of completeness theorem in $Ł_\infty$. However, even though we have the completeness for the set of tautologies both in classical propositional logic and in Łukasiewicz propositional logic, things change if we refer to the *strong completeness*. Given a nonempty set of formulas Θ we can refer to deductive closure of Θ in two different ways. We say that a formula ϕ is a *semantic consequence* of Θ, in symbols $\Theta \models \phi$, if for each valuation v such that $v(\theta) = 1$, for all $\theta \in \Theta$ then $v(\phi) = 1$ (this is the Bolzano-Tarski paradigm). On the other hand, ϕ is a *syntactic consequence* of Θ, in symbols $\Theta \vdash \phi$, if there is a proof of ϕ from Θ. In classical logic the two sets of semantic and syntactic consequences coincide, while in $Ł_\infty$, as we shall see, these two sets do not coincide in general. The reason why we do not have the strong completeness in $Ł_\infty$ lies in the notion of semantic consequence which turns out to be unsuitable. In fact when we consider the semantic consequences of a set Θ of formulas we refer to valuations which can be seen has homomorphism from the Lindenbaum algebra L of Ł$_\infty$ to the MV-algebra [0,1]. In particular there is a one-one correspondence between the set of all valuations and the set of maximal ideals of L and each valuation can be seen as a quotient of L respect to a maximal ideal M. In Boolean algebras, the maximal ideals have the property of being irreducible, while maximal ideals of MV-algebras do not have this property which is crucial for completeness. Suppose that $\Theta \vdash \phi$, with Θ nonempty set of formulas, then the class $[\phi]/I(\Theta) = 1$ in the quotient $L/I(\Theta)$, where $I(\Theta)$ is the ideal of L generated by $[\Theta]$. Denoting with A the quotient $L/I(\Theta)$ and with $\mathcal{M}(A)$ the set of its maximal ideals, we can consider the family of quotients $\{A/M \mid M \in \mathcal{M}(A)\}$ and the homomorphism

$$\beta\colon a = [\alpha]/I(\Theta) \in A \to (a/M \mid M \in \mathcal{M}(A)) \in \prod_{M \in \mathcal{M}(A)} A/M$$

Requiring completeness is equivalent to requiring that the map β is injective which is equivalent to ask that $I(\Theta) = \bigcap_{\bar{M} \subseteq I(\Theta)} \bar{M}$ with $\bar{M}$ maximal ideal of L. But whenever we consider a proper ideal J of an MV-algebra, it is not always true that it coincides with the intersection of all maximal ideals which contains it. In Boolean algebras this property is satisfied by the maximal ideals since they coincide with the irreducible ideals. Moreover, maximal ideals in MV-algebras are 'too big' and, therefore, they give 'too small' quotients. This entails a loss of information in a sense that will be clarified. Therefore, we want to find a family of ideals which gives us bigger quotients and could give us a new notion of valuation that turns out to be strongly complete. As we shall see, the *prime ideals* of any MV-algebra are irreducible, i.e. each proper ideal of A is the intersection of all prime ideals which contain it. Thus, through the study and the characterization of prime ideals of the Lindenbaum algebra L of Ł$_\infty$ we shall give new enriched notions of valuation and semantic consequence which could give us the completeness in the strong sense.

The thesis is structured as follows.

In Chapter 1 we give some necessary basic notions concerning MV-algebras, in particular we give an introduction to free MV-algebras which will be characterized in the subsequent chapter. Then we give some basic notions about Łukasiewicz propositional calculus and the connections between the Lindenbaum algebra and the theories.

Thanks to McNaughton theorem the free MV-algebras over k-generators can be seen as the MV-algebra of McNaughton functions M_k given by piecewise linear functions with integer coefficients defined over $[0,1]^k$ with values in $[0,1]$. In Chapter 2 we study the ideals of the MV-algebra of McNaughton functions M given by all McNaughton function defined in $[0,1]^\omega$, giving some results concerning quotients of M. In particular we give a geometrical characterization for prime ideals of M_n, with $n \in \mathbb{N}$.

In Chapter 3 we face the problem of completeness. First of all we give an algebraic interpretation of the sets of semantic and syntactic consequences of a set Θ of formulas in order to give some necessary and sufficient conditions for these two sets to coincide. What we find is that the completeness theorem is satisfied if and only if the Lindenbaum algebra of Θ is semisimple which is equivalent to say that the ideal of the Lindenbaum algebra L generated by $[\Theta]$ coincides with the intersection of all maximal ideals which contain it. Subsequently we give the notion of *differential valuations* which can be seen as an evolution of usual valuations. These new valuations are linked with prime ideals and through them we give a new notion of semantic consequence which satisfies the strong completeness theorem.

Chapter 1

Preliminary notions

The following chapter is useful to be in touch with the argument we are treating. It is known that classical logic gives rise to the study of Boolean algebras, similarly MV-algebras are the *algebraic semantics* of Łukasiewicz many-valued logic, as a matter of fact the letters 'MV' stand for many-valued logic.

Thus, in this chapter some basic notions about MV-algebras, concerning both the arithmetics and the structure of these algebras, are explained. In particular we shall introduce the free MV-algebras whose study will be deepened in the next chapter giving an important characterization for these particular MV-algebras. Moreover, in the last sections, we shall give an introduction to Łukasiewicz propositional calculus Ł_∞ reserving a particular interest for the Lindenbaum Algebra and for the theories.

For all the unexplained notions we refer to [3], [2] and [8].

1.1 MV-algebras

Definition 1.1.1. An MV-algebra is an algebra $\langle A, \oplus, \neg, 0 \rangle$ with a binary operation $\oplus$, a unary operation $\neg$ and a constant 0 satisfying the following equations:

MV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$

MV2) $x \oplus y = y \oplus x$

MV3) $x \oplus 0 = x$

MV4) $\neg\neg x = x$

MV5) $x \oplus \neg 0 = \neg 0$

MV6) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$

In particular, axioms MV1)-MV3) state that $\langle A, \oplus, \neg, 0\rangle$ is an *abelian monoid*. The singleton $\{0\}$ is a trivial example of MV-algebra. An MV-Algebra is *nontriavial* if and only if its universe has more than one element. We shall denote any MV-algebra $\langle A, \oplus, \neg, 0\,\rangle$ with its universe A.

Example 1.1.2. The real unit interval [0,1] with the following operations

$$x \oplus y =_{def} \min(1, x+y)$$

$$\neg x =_{def} 1 - x$$

is an MV-algebra, denoted by [0,1].

Example 1.1.3. If $\langle A, \vee, \wedge, -, 0\rangle$ is a Boolean algebra, then $\langle A, \vee, -, 0\rangle$ is an MV-algebra where $\vee$, $-$ and 0 denote the joint, the complement and the smallest element in A, respectively.

Example 1.1.4. Given an MV-algebra A and a non-empty set X, the set A^X of all functions $f\colon X \to A$ is an MV-algebra with the operations $\oplus$ and $\neg$ defined pointwise as follows

$$(f \oplus g)(x) = f(x) \oplus g(x)$$

$$(\neg f)(x) = \neg f(x)$$

In any MV-algebra A we define the constant 1 and the operations $\odot$ and $\ominus$ as follows:

1. $1 =_{def} \neg 0$
2. $x \odot y =_{def} \neg(\neg x \oplus \neg y)$
3. $x \ominus y =_{def} x \odot \neg y$

Recalling the example 1.1.2, in the MV-algebra [0,1] we have $x \odot y = max(0, x+y-1)$ and $x \ominus y = max(0, x-y)$. Thus, an MV-algebra is nontrivial if and only if $0 \neq 1$ and the following identities hold for every $x, y \in A$:

MV7) $\neg 1 = 0$

MV8) $x \oplus y = \neg(\neg x \odot \neg y)$

MV9) $x \oplus \neg x = 1$

Axioms MV5) and MV6) can be written as:

MV5') $x \oplus 1 = 1$

MV6') $(x \ominus y) \oplus y = (y \ominus x) \oplus x$

Lemma 1.1.5. *Considering an MV-algebra A, for any two elements $x, y \in A$ the following are equivalent*

(i) $\neg x \oplus y = 1$

(ii) $x \odot \neg y = 0$

(iii) $y = x \oplus (y \ominus x)$

(iv) there is an element $z \in A$ such that $x \oplus z = y$

Proof. $(i) \to (ii)$ It follows from axioms MV4) and MV7).
$(ii) \to (iii)$ By MV3) and MV6').
$(iii) \to (iv)$ It is sufficient to take $z = y \ominus x$.
$(iv) \to (i)$ By MV9), $\neg x \oplus x \oplus z = 1$. □

For any two elements $x, y \in A$, we can defined the relation

$$x \leq y$$

saying that $x \leq y$ if and only if x and y satisfy the equivalent conditions of lemma 1.1.5. One can easily observe that $\leq$ is a partial order, called the natural order of A: the reflexivity is equivalent to MV9), antisymmetry follows from conditions (ii) and (iii) of lemma 1.1.5, and transitivity follows from conditions (iv).

Definition 1.1.6. An MV-algebra whose natural order is total is called an *MV-chain.*

Remark 1.1.7. Note that, by lemma 1.1.5*(iv)*, the order of the MV-chain [0,1] coincides with the natural order of the real numbers.

Lemma 1.1.8. *Let A be an MV-algebra. For each $a \in A$, $\neg a$ is the unique solution of the simultaneous equations:*

$$\begin{cases} a \oplus x = 1 \\ a \odot x = 0 \end{cases}$$

Proof. By lemma 1.1.5, from these two equations $\neg a \leq x$ and $\neg a \geq x$. Thus, $x = \neg a$. ☐

Lemma 1.1.9. *In every MV-algebra A the natural order $\leq$ has the following properties:*

(i) $x \leq y$ *iff* $\neg y \leq \neg x$

(ii) *if* $x \leq y$ *then for each* $z \in A$, $x \oplus z \leq y \oplus z$ *and* $x \odot z \leq y \odot z$

(iii) $x \odot y \leq z$ *iff* $x \leq \neg y \oplus z$

Proof. (i). It follows from lemma 1.1.5(i), since $\neg x \oplus y = \neg\neg y \oplus \neg x$.
(ii). The monotonicity of $\oplus$ is an immediate consequence of lemma 1.1.5(iv); using (i) it is easy to check the monotonicity of $\odot$.
(iii). It is sufficient to note that $x \odot y \leq z$ is equivalent to $1 = \neg(x \odot y) \oplus z = \neg x \oplus \neg y \oplus z$. ☐

Proposition 1.1.10. *On each MV-algebra A the natural order determines a lattice structure. Specifically, the join $x \vee y$ and the meet $x \wedge y$ of the elements x and y are given by:*

(i) $x \vee y = (x \odot \neg y) \oplus y = (x \ominus y) \oplus y$

(ii) $x \wedge y = \neg(\neg x \vee \neg y) = x \ominus (\neg y \oplus y)$

Proof. To prove (i), by MV6'), MV9) and lemma 1.1.9(ii), we have:

$$x \leq (x \ominus y) \oplus y$$

$$y \leq (x \ominus y) \oplus y.$$

Suppose that $x \leq z$ and $y \leq z$. Then, by lemma 1.1.5, $\neg x \oplus z = 1$ and $z = (z \ominus y) \oplus y$. Thus by MV6'):

$$\begin{aligned}(\neg(x \ominus y) \oplus y) \oplus z &= (\neg(x \ominus y) \ominus y) \oplus y \oplus (z \ominus y) \\ &= (y \ominus \neg(x \ominus y)) \oplus \neg(x \ominus y) \oplus (z \ominus y) \\ &= (y \ominus \neg(x \ominus y)) \oplus \neg x \oplus y \oplus (z \ominus y) \\ &= (y \ominus \neg(x \ominus y)) \oplus \neg x \oplus z = 1\end{aligned}$$

It follows that $(x \ominus y) \oplus y \leq z$, which completes the proof of (i). Condition (ii) is a consequence of (i) combined with lemma 1.1.9. □

Proposition 1.1.11. *The following equations hold in every MV-algebra:*

(i) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$

(ii) $x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$

Proof. By MV6') and lemma 1.1.9(ii), $x \odot y \leq x \odot (y \vee z)$ and $x \odot z \leq x \odot (y \vee z)$. Suppose that $x \odot y \leq t$ and $x \odot z \leq t$. Then by lemma 1.1.9(iii), $y \leq \neg x \odot t$ and $z \leq \neg x \oplus t$, whence $y \vee z \leq \neg x \oplus t$. One more application of lemma 1.1.9(iii) yields $(y \vee z) \odot x \leq t$, which completes the proof of (i). It is easy to see that (ii) is a consequence of (i) using lemma 1.1.9(i), together with MV4) and MV8). □

Definition 1.1.12. Let A be an MV-algebra. For each $x \in A$ and each integer $n \geq 0$

$$0x = 0$$

$$(n+1)x = nx \oplus x$$

Lemma 1.1.13. *Let x and y be elements of an MV-algebra A. If $x \wedge y = 0$ then for each integer $n \geq 0$, $nx \wedge ny = 0$.*

Proof. Suppose that $x \wedge y = 0$. By monotonicity (lemma 1.1.9) and distributivity of $\wedge$ (proposition 1.1.11), we obtain $x = x \oplus (x \wedge y) = (x \oplus x) \wedge (x \oplus y) \geq 2x \wedge 2y$, whence $0 = x \wedge y \geq 2x \wedge 2y$. It follows that $0 = 2x \wedge 2y = 4x \wedge 4y = 8x \wedge 8y = \dots$. The conclusion follows from $nx \wedge ny \leq 2^n x \wedge 2^n y = 0$. □

Definition 1.1.14. A *subalgebra* of an MV-algebra is a subset B of A containing the zero element of A and closed under the operations of A.

Recalling the example 1.1.2, for each integer $n \geq 2$, the n-element sets:

$$\mathbf{L}_n =_{def} \{0,\ 1/(n-1),\ \ldots,\ (n-2)/(n-2),\ 1\}$$

yield examples of subalgebras of [0,1].
Recalling the example 1.1.4, the set of continuous functions from [0,1] into [0,1] forms a subalgebra of the MV-algebra $[0,1]^{[0,1]}$.

Definition 1.1.15. Let A and B be MV-algebras. A function $h : A \to B$ is a homomorphism if and only if, for each $x, y \in A$, it satisfies the following conditions:

(i) $h(0) = 0$

(ii) $h(x \oplus y) = h(x) \oplus h(y)$

(iii) $h(\neg x) = \neg h(x)$

If h is one-one we say that h is a monomorphism. If $h\colon A \to B$ is onto B we say that h is surjective. By *isomorphism* we shall mean a surjective one-one homomorphism. If there is an isomorphism from A onto B, we write $A \cong B$.

Definition 1.1.16. Given a homomorphism $h : A \to B$, the *kernel* of h is defined as follows

$$Ker(h) =_{def} h^{-1}(0) = \{a \in A \mid h(a) = 0\}$$

Definition 1.1.17. A set $I \subseteq A$ is an ideal iff the following conditions are satisfied:

(i) $0 \in I$

(ii) if $x, y \in I$ then $x \oplus y \in I$

(iii) if $x \in I$ and $y \in A$, with $y \leq x$ then $y \in I$

The intersection of any family of ideals of A is still an ideal of A. Let W be a generic subset of A, the intersection of all ideals $J \supseteq W$ of A is always an ideal and is called *the ideal* *generated* by W and is denoted by $\langle W \rangle$. An ideal I is *proper* iff $I \neq A$; I is *prime* iff is proper and given $x, y \in A$ either $x \ominus y \in I$ or $y \ominus x \in I$; I is *maximal* iff is proper and there is no proper ideal $J \subset A$ such that $I \subset J$, i.e for each ideal $J \neq I$ such that $I \subset J$ then $J = A$. We denote with $\mathbf{I}(A), \mathbf{P}(A)$ and $\mathbf{M}(A)$ the sets of ideals, prime ideals and maximal ideals of A, respectively.

Lemma 1.1.18. *Let W ba a subset of an MV-algebra A. If $W = \emptyset$ then $\langle W \rangle = \{0\}$. If $W \neq \emptyset$, then*

$$\langle W \rangle = \{x \in A \mid x \leq w_1 \oplus \cdots \oplus w_k, \ \textit{for some}\ w_1, \dots w_k \in W\}$$

For each element $z \in A$, the ideal $\langle z \rangle = \langle \{z\} \rangle = \{x \in A \mid nz \geq x \ \textit{for some integer}\ n \geq 0\}$ is called the principal ideal generated by z and, for each $a \in A$ and J ideal of A:

$$\langle J \cup \{z\} \rangle = \{x \in A \mid x \leq nz \oplus a, \ \textit{for some}\ n \in \mathbb{N} \ \textit{and}\ a \in J\}$$

Proposition 1.1.19. *For any proper ideal J of an MV-algebra A the following conditions are equivalent*

(i) J is a maximal ideal of A

(ii) for each $x \in A$, $x \notin J$ iff $\neg nx \in J$ for some integer $n \geq 1$.

Proof. $(i) \rightarrow (ii)$. Suppose that J is a maximal ideal of A. If $x \notin J$, then $\langle \{x\} \cup J \rangle = A$ and, by lemma 1.1.18, $1 = nx \oplus a$ for some integer $n \geq 1$ and $a \in J$. Then by lemma 1.1.5 $\neg xn \leq a \in J$ whence by definition of ideal, $\neg nx \in J$. Conversely, if $x \in J$, then $nx \in J$ for each integer $n \geq 1$; since J is proper $\neg nx \notin J$.

$(ii) \rightarrow (i)$. Let $K \neq J$ be an ideal of A such that $J \subseteq K$. For every $x \in K \setminus J$ it follows, from the hypothesis, that $\neg nx \in J$ for some integer $n \geq 1$. Hence $1 = nx \oplus \neg nx \in K$ and $K = A$. ◻

Lemma 1.1.20. *Let A, B be MV-algebras, and $h : A \rightarrow B$ a homomorphism. Then the following properties hold:*

(i) For each ideal J of B, the set $h^{-1}(J) =_{def} \{x \in A \mid h(x) \in J\}$ is an ideal of A. Thus in particular, Ker(h) is an ideal of A.

(ii) $h(x) \leq h(y)$ *iff* $x \ominus y \in Ker(h)$

(iii) h *is injective iff* $Ker(h) = 0$

(iv) $Ker(h) \neq A$ *iff* B *is nontrivial*

(v) $Ker(h)$ *is a prime ideal of* A *iff* B *is nontrivial and the image* $h(A)$*, as a subalgebra of* B*, is an MV-chain.*

Proposition 1.1.21. *Let* I *be an ideal of an* MV*-algebra* A*. Then the binary relation* $\equiv_I$ *on* A *defined by, for each* $x, y \in A$*:*

$$x \equiv_I y \text{ iff } (x \ominus y) \oplus (y \ominus x) \in I$$

is a congruence relation, i.e. $\equiv_I$ *is an equivalence relation such that* $x \equiv_I y$ *and* $t \equiv_I z$ *imply* $\neg x \equiv_I \neg y$ *and* $x \oplus t \equiv_I y \oplus z$*. Moreover* $I = \{x \in A \mid x \equiv_I 0\}$ *Conversely if* $\equiv$ *is a congruence on* A*, then* $\{x \in A \mid x \equiv 0\}$ *is an ideal, and* $x \equiv y$ *iff* $(x \ominus y) \oplus (y \ominus x) \equiv 0$*. Therefore, the correspondence* $I \mapsto \equiv_I$ *is a bijection from the set of ideals of* A *onto the set of congruences on* A*.*

Given $x \in A$, we denote with x/I the equivalence class of x respect to $\equiv_I$ and with A/I the quotient set $A/{\equiv_I}$. Since $\equiv_I$ is a congruence, the set A/I inherits the structure of MV-algebra from A defining the following operations:

$$\neg(x/I) =_{def} (\neg x/I)$$

$$x/I \oplus y/I =_{def} (x \oplus y)/I$$

We denote with $\langle A/I, \oplus, \neg, 0/I \rangle$ the quotient of A by the ideal I, then the correspondence $x \to x/I$ defines a surjective homomorphism h_I called the natural homomorphism from A onto the quotient A/I. Note that $Ker(h_I) = I$. The next lemma is a consequence of lemma 1.1.20.

Lemma 1.1.22. *If* A*,* B *and* C *are* MV*-algebras, and* $f\colon A \to B$ *and* $g\colon A \to C$ *are surjective homomorphisms, then* $Ker(f) \subseteq Ker(g)$ *if and only if there is a surjective homomorphism* $h\colon B \to C$ *such that* $h \circ f = g$*. This homomorphism* h *is an isomorphism if and only if* $Ker(f) = Ker(g)$*.*

Theorem 1.1.23. *Let A and B be MV-algebras. If $h\colon A \to B$ is a surjective homomorphism, then there is an isomorphism $f\colon A/Ker(h) \to B$ such that $f(x/Ker(h)) = h(x)$ for all $x \in A$.*

Proposition 1.1.24. *If an MV-algebra A is a MV-chain then all proper ideals of A are prime.*

Proof. Let I be a proper ideal of A. Since $h_I\colon A \to A/I$ is a surjective homomorphism and A is a MV-chain, A/I is an MV-chain. Whence, by lemma 1.1.20(v), I must be a prime ideal of A. □

Proposition 1.1.25. *Let J be an ideal of an MV-algebra A. Then the map $I \to h_J(I)$ determines an inclusion preserving one-one correspondence between the ideals of A containing J and the ideals of the quotient MV-algebra A/J. The inverse map also preserves inclusions and is obtained by taking the inverse image ${h_J}^{-1}(K)$ of each ideal K of A/J.*

Proof. Let I be an ideal of A such that $J \subseteq I$. Since h_J maps A onto A/J and $Ker(h_J) = J \subseteq I$, by lemma 1.1.20(ii) and MV6'), it follows that $h_J(I) \in \mathbf{I}(A/J)$ and ${h_J}^{-1}(h_J(I)) \subseteq I$. Since the converse inclusion holds for all surjective mappings, then $I = {h_J}^{-1}(h_J(I))$. On the other hand, by lemma 1.1.20(i), ${h_J}^{-1}(K) \in \mathbf{I}(A)$ for each $K \in \mathbf{I}(A/J)$. To complete the proof it is sufficient to observe that $J = {h_J}^{-1}(\{0\}) \subseteq {h_J}^{-1}(K)$ and $h_J({h_J}^{-1}(K)) = K$. □

If A is an MV-chain, then the set $\mathbf{I}(A)$ of ideals of A is totally ordered by inclusion. Indeed, if I and J were ideals of A such that $I \nsubseteq J$ and $J \nsubseteq I$ then there would be elements $a, b \in A$ such that $a \in I/J$ and $b \in J/I$ whence $a \nleq b$ and $b \nleq a$, which is impossible.

Theorem 1.1.26.

(i) *Every proper ideal J of an MV-algebra A that contains a prime ideal is prime.*

(ii) *For each prime ideal J of A, the set $\{I \in \mathbf{I}(A) \mid J \subseteq I\}$ is totally ordered by inclusion.*

Proof. Let J be a prime ideal of A, by lemma 1.1.20(v), the quotient A/J is an MV-chain thus, by proposition 1.1.24, all proper ideals of A/J are prime and are totally ordered by inclusion. This, together with proposition 1.1.25, implies (*ii*). In order to prove (*i*), let us note that if I is a proper ideal of A such that $J \subseteq I$ and, again by proposition 1.1.25, $I = {h_J}^{-1}(h_J(I))$, hence I is a prime ideal of A/J. $\square$

Corollary 1.1.27. *Every prime ideal J of an MV-algebra A is contained in a unique maximal ideal of A.*

Proof. Consider the set

$$\mathbf{H} =_{def} \{I \in \mathbf{I}(A) \mid I \neq A \text{ and } J \subseteq I\}$$

Since $\mathbf{H}$ is totally ordered by inclusion, the set $M = \cup_{I \in \mathbf{H}} I$ is an ideal of A. M is also a proper ideal of A because $1 \notin M$. Suppose that there exists K proper ideal of A such that $M \subseteq K$, therefore $J \subseteq K$ and $K \in \mathbf{H}$. Hence: $K \subseteq \cup_{I \in \mathbf{H}} I = M \subseteq K$. This implies $K = M$, so M is the only maximal ideal containing J. $\square$

Lemma 1.1.28. *For every MV-algebra A and ideal $J \neq A$ the following conditions are equivalent:*

(i) *J is prime;*

(ii) *for all $x, y \in A$ if $x \wedge y = 0$ then $x \in J$ or $y \in J$;*

(iii) *for all $x, y \in A$ if $x \wedge y \in J$ then $x \in J$ or $y \in J$;*

(iv) *if P and Q are ideals of A and $P \cap Q \subseteq J$ then $P \subseteq J$ or $Q \subseteq J$;*

(v) *if P and Q are ideals of A and $P \cap Q = J$ then $P = J$ or $Q = J$;*

(vi) *if P and Q are ideals of A containing J then $P \subseteq Q$ or $Q \subseteq P$;*

(vii) *for all $x, y \in A$ either $x \to y \in J^*$ or $y \to x \in J^*$, where J^* is the filter given by the set $\{\neg z \mid z \in J\}$;*

(viii) *for all $x, y \in A$ either $x \ominus y \in J$ or $y \ominus x \in J$.*

The following proposition plays an important role in the proof of *Chang's Subdirect Representation Theorem* 1.2.3.

Proposition 1.1.29. *Let A be an MV-algebra, J an ideal of A and $a \in A \setminus J$. Then there is a prime ideal P of A such that $J \subseteq P$ and $a \notin P$.*

Proof. By an application of Zorn's Lemma it is possible to show that there is an ideal I of A such that I is maximal with respect to the property that $J \subseteq I$ and $a \notin I$. In order to show that I is a prime ideal, let x and y be element of A and suppose that both $x \ominus y \notin I$ and $y \ominus x \notin I$ (absurdum hypothesis). Then the ideal generated by I and $x \ominus y$ must contain the element a. By lemma 1.1.18, $a \leq s \oplus p(x \ominus y)$ for some $s \in I$ and some integer $p \geq 1$. Similarly, there is an element $t \in I$ and an integer $q \geq 1$ such that $a \leq t \oplus q(y \ominus x)$. Let $u = s \oplus t$ and $n = \max(p, q)$. Then $u \in I$, $a \leq u \oplus n(x \ominus y)$ and $a \leq u \oplus (y \ominus x)$. It follows that $a \leq (u \oplus n(x \ominus y)) \wedge (u \oplus n(y \ominus x)) = u \oplus (n(x \ominus y) \wedge n(y \ominus x)) = u$. Hence $a \in I$, a contradiction. □

Corollary 1.1.30. *Every proper ideal of an MV-algebra is an intersection of prime ideals.*

Corollary 1.1.31. *Every nontrivial MV-algebra has a maximal ideal.*

1.2 Subdirect representation of MV-algebras

The *direct product* of family $\{A_i\}_{i \in I}$ of MV-algebras, where I denotes a nonempty set, is the MV-algebra, denoted with $\prod_{i \in I} A_i$, obtained by defining pointwise MV-operations on the set-theoretical cartesian product of the family. In other words, $\prod_{i \in I} A_i$ is the space of the functions $f : I \to \bigcup_{i \in I} A_i$ such that $f(i) \in A_i$ for all $i \in I$, with the two operations $\neg$ and $\oplus$ defined as follows:

$$(\neg f)(i) =_{def} \neg f(i) \qquad (f \oplus g)(i) =_{def} (f(i) \oplus g(i))$$

The zero element of $\prod_{i \in I} A_i$ is the function $i \in I \to 0_i \in A_i$.

For each $j \in I$ it is possible to define a homomorphism onto A_j:

$$\pi_j : \prod_{i \in I} A_i \to A_j \text{ such that } \pi_j(f) =_{def} f(j)$$

This homomorphism is called the j^{th} *projection function.* In particular, for each MV-algebra A and nonempty set X, the MV-algebra A^X is the direct product of the family $\{A_x\}_{x \in X}$ with $A_x = A$ for all $x \in X$.

Definition 1.2.1. An MV-algebra A is a subdirect product of a family $\{A_i\}_{i\in I}$ of MV-algebras iff there exists a one-one homomorphism $h : A \to \prod_{i\in I} A_i$ such that for each $j \in I$, the composite map $\pi_j \circ h$ is a homomorphism onto A_j.

In other words if A is a subdirect product of the family $\{A_i\}_{i\in I}$, then A is isomorphic to the subalgebra $h(A)$ of $\prod_{i\in I} A_i$ and, moreover, the restriction to $h(A)$ of each projection is a surjective mapping. The following result is a particular case of *Birkhoff's Theorem.*

Theorem 1.2.2. *An MV-algebra A is a subdirect product of a family $\{A_i\}_{i\in I}$ iff there is a family $\{J_i\}_{i\in I}$ of ideals of A such that*

(i) $A_i \cong A/J_i$ *for each* $i \in I$

(ii) $\bigcap_{i\in I} J_i = \{0\}$

Proof. Suppose that A is a subdirect product of a family $\{A_i\}_{i\in I}$ of MV-algebras, let h be a one-one homomorphism given in definition 1.2.1. Consider $J_j := Ker(\pi_j \circ h)$. By theorem 1.1.23, $A_j \cong A/J_j$. If $x \in \bigcap_{i\in I} J_i$ then $\pi_j(h(x)) = 0$ for all j. Then $h(x) = 0$, and since h is injective $x = 0$. Therefore $\bigcap_{i\in I} J_i = \{0\}$.
Conversely, suppose that $\{J_i\}_{i\in I}$ is a family of ideals of A satisfying condition (i) and (ii). Let ϵ_i be the isomorphism given by condition (i). Let h be the function

$$h\colon A \to \prod_{i\in I} A_i$$

defined by stipulating that, for each $x \in A$

$$(h(x))(i) = \epsilon_i(x/J_i)$$

by condition (ii), it follows that $Ker(h) = \{0\}$, whence by lemma 1.1.20(iii), h is injective. Since for each $i \in I$ the map $a \in A \mapsto a/J_i \in A/J_i$ is surjective, then $pi_i \circ h$ maps A onto A_i. Thus, A is a subdirect product of the family $\{A_i\}_{i\in I}$. □

Theorem 1.2.3 (Chang's Subdirect Representation Theorem)**.** *Every nontrivial MV-algebra is a subdirect product of MV-chains.*

Proof. By theorem 1.2.2 and lemma 1.1.20(v), an MV-algebra A is a subdirect product of a family of MV-chains if and only if there is a family $\{P_i\}_{i\in I}$ of prime ideals of A such that $\bigcap_{i\in I} P_i = \{0\}$. Now it is sufficient to apply corollary 1.1.30 to the ideal $\{0\}$. □

1.3 MV-equations

An important consequence of Chang's Subdirect Representation Theorem is that to verify whether an equation holds in all MV-algebras it is sufficient to check that the equation holds in all MV-chains.

Remark 1.3.1. An MV-equation $\tau = \sigma$ holds in an MV-algebra A if and only if the equation $(\tau \ominus \sigma) \oplus (\sigma \ominus \tau) = 0$ holds in A. Therefore it can be assumed that all the MV-equation are of the form $\rho = 0$, where ρ is an MV-term.

Lemma 1.3.2. *Let A, B, A_i (for $i \in I$) be MV-algebras:*

(i) if $A \models \tau = \sigma$ then $S \models \tau = \sigma$ for each subalgebra S of A

(ii) if $h : A \to B$ is a homomorphism, then for each MV-term τ in the variables $x_i, \ldots, x_n$ and each n-uple $a_1, \ldots, a_n$ of elements of A we have $h(\tau^A(a_1, \ldots, a_n)) = \tau^B(h(a_1), \ldots, h(a_n))$. In particular, when h is surjective, from $A \models \tau = \sigma$ it follows $B \models \tau = \sigma$

(iii) if $A_i \models \tau = \sigma$ for each $i \in I$ then $\prod_{i \in I} A_i \models \tau = \sigma$

Theorem 1.3.3. *Let A be a subdirect product of a family $\{A_i\}_{i \in I}$ of MV-algebras, let $\tau = \sigma$ be an MV-equation. Then $A \models \tau = \sigma$ if and only if $A_i \models \tau = \sigma$ for each $i \in I$.*

Corollary 1.3.4. *An MV-equation is satisfied by all MV-algebras if and only if it is satisfied by all MV-chains.*

Corollary 1.3.4 becomes more significant in the light of *Chang's Completeness Theorem.*

Theorem 1.3.5 (Chang's Completeness Theorem)**.** *An equation holds in* [0,1] *if and only if it holds in every MV-algebra.*

Proof. See [3, 2.5.3] for details. □

Thus, intuitevely, the two element structure {0,1} stands to Boolean algebras as the interval [0,1] stands to MV-algebras.

1.4 Free MV-algebras

Free MV-algebras are 'universal' objects: it can be shown that every finitely generated MV-algebra is isomorphic to a quotient of the free MV-algebra over n generators. Moreover, another important property is that every equation satisfied in the free MV-algebra ω-generated is also satisfied in every MV-algebra.
Let k be an arbitrary cardinal ≥ 1 and consider k distinct propositional variables

$$X_\alpha \text{ with } \alpha < k \tag{1.1}$$

Then each MV-term τ in the variables $\{X_\alpha\}_{\alpha<k}$ is a finite string of symbols over the alphabet

$$\{0,\, \neg,\, \oplus,\, (,\,),\, X_\alpha\}_{\alpha<k} \tag{1.2}$$

For any MV-algebra A the elements of A^k have the form

$$\bar{a} = (a_\alpha \mid \alpha < k) \tag{1.3}$$

where each a_α with $\alpha < k$ is an element of A. We call α^{th} projection the map $\pi_\alpha \colon A^k \to A$ such that $\pi_\alpha(\bar{a}) = a_\alpha$. The set of projections of A^k is denoted by:

$$Proj_k{}^A =_{def} \{\pi_\alpha \mid \alpha < k\} \tag{1.4}$$

Definition 1.4.1. For each term τ in the variables $\{X_\alpha\}_{\alpha<k}$ the term function

$$\tau^A \colon A^k \to A \tag{1.5}$$

is given by induction on the complexity of τ as follows:

1. $X_\alpha{}^A =_{def} \pi_\alpha$
2. 0^A is the constant function 0 on A^k
3. $(\neg\rho)^A =_{def} \neg^A(\rho^A)$
4. $(\rho \oplus \sigma)^A =_{def} (\rho^A \oplus^A \sigma^A)$

The set of all term functions over A^k is denoted by ${Term_k}^A$. The operations $\neg^A$ and $\oplus^A$ are defined pointwise as in example 1.1.4, thus ${Term_k}^A$ is a subalgebra of the MV-algebra A^{A^k}.

Remark 1.4.2. Each element of ${Term_k}^A$ is a function depending on a finite number of variables.

Lemma 1.4.3. *For each MV-algebra A and for each cardinal $k \geq 1$, ${Term_k}^A$ is the smallest subalgebra of A^{A^k} containing ${Proj_k}^A$.*

Definition 1.4.4. Let A be an MV-algebra and Y be a subset of elements generating A, A is said to be free over Y and is denoted with $Free_Y$ iff for every MV-algebra B and every function $f\colon Y \to B$, f can be extended to a unique homomorphism $F\colon A \to B$.

Remark 1.4.5. For any two sets Y and X of the same cardinality k, if A is free over Y and B is free over X then $A \cong B$. Therefore, without ambiguity we can call A "the" free MV-algebra over k many generators and we can write $A = Free_k$.

Proposition 1.4.6. *For each cardinal $k \geq 1$, ${Term_k}^{[0,1]}$ is the free MV-algebra over ${Proj_k}^{[0,1]}$, in symbols ${Term_k}^{[0,1]} \cong Free_k$.*

Proof. Let B be an MV-algebra and $f\colon {Proj_k}^{[0,1]} \to B$ be a function. Let $b = (b_0, \dots, b_\alpha, \dots)_{\alpha \leq k}$ be the element of B^k given by: $b_\alpha = f(\pi_\alpha)$. We define a function ϕ that maps each term τ, in the variables X_α with $\alpha < k$, into the element $\tau^B(b) \in B$, where $\tau^B \in Term_k^B$ is the term function evaluated in B^k, determined by τ. By theorem 1.3.5 it follows that:

$$[0,1] \models \tau = \rho \implies B \models \tau = \rho$$

thus,

$$\rho^B(b) = \tau^B(b)$$

In other words, if $\rho^{[0,1]} = \tau^{[0,1]} \in Term_k^{[0,1]}$ then ϕ maps ρ and τ in the same element in B. Since $\phi(X_\alpha) = X_\alpha^B(b) = b_\alpha = f(\pi_\alpha)$, the function ϕ determines an extension F of f. By induction, it is easy to check that F is an homomorphism. In order to prove the uniqueness, let $g : Term_k^{[0,1]} \to B$ be an homomorphism extending f. Since F and g coincide over a subalgebra of $[0,1]^{[0,1]^k}$ containing projections then by lemma 1.4.3 they coincide over all $Term_k^{[0,1]}$. □

The following proposition is an immediate consequence of lemma 1.1.20 and theorem 1.1.23.

Proposition 1.4.7. *Let $k \geq 1$ be a cardinal and let A be an MV-algebra generated by $\leq k$ elements. Then there is an ideal J of the $Free_k$ such that A is isomorphic to the quotient algebra $Free_k/J$.*

In the next chapter we will give a more explicit description of elements of the algebras $Free_k$, introducing McNaughton functions and some of their properties.

1.5 An introduction to Łukasiewicz propositional calculus Ł$_\infty$

In the Łukasiewicz propositional calculus Ł$_\infty$, the negation $\neg$ and implication $\rightarrow$ are considered as the main connectives. Through them it is possible to define the other Łukasiewicz connectives $\odot$ and $\oplus$ of conjuction and disjunction as follows

$$\alpha \oplus \beta =_{def} \neg\alpha \rightarrow \beta$$

$$\alpha \odot \beta =_{def} \neg(\neg\alpha \oplus \neg\beta)$$

The set of propositional formulas is defined as in the Boolean case, from a denumerable set of propositional variables $Var = \{X_0, X_1, \ldots, X_n, \ldots\}$, through the connectives $\neg$ and $\rightarrow$. We denote with $FORM$ the set of all formulas.

Definition 1.5.1. The set $FORM$ is given inductively as follows:

(i) Each propositional variable X_k is a formula

(ii) If α is a formula, then $(\neg\alpha)$ is a formula

(iii) If α and β are formulas, then $(\alpha \rightarrow \beta)$ is a formula

We shall denote with $FORM_n$ the set of all formulas built from a finite subset of $n \geq 1$ variables in Var.

Definition 1.5.2. Let A be an MV-algebra. Then an A-valuation is a function

$$v\colon FORM \to A$$

satisfying the following properties, with α and β formulas:

(i) $v(\neg\alpha) =_{def} \neg v(\alpha)$

(ii) $v(\alpha \to \beta) =_{def} v(\alpha) \to v(\beta)$

Any A-valuation is uniquely determined by its values

$$v(X_0), \dots, v(X_n), \dots$$

Given an A-valuation v, we say that v satisfies the formula α iff $v(\alpha) = 1$; a formula α is a *tautology* if and only if α is satisfied by all A-valuations. Let α and β be formulas, then α and β are semantically A-equivalent iff $v(\alpha) = v(\beta)$ for all A-valuations v. Given $\Theta \subseteq FORM$, a formula α is a semantic A-consequence of Θ if and only if each A-valuation v that satisfies all formulas in Θ also satisfies α. Therefore α is an A-tautology if and only if α is a semantic A-consequence of the empty set.

Every formula α containing the variables $X_1, \dots, X_k$ can be transformed into an MV-term τ_α in the same variables. Conversely, upon replacing every occurrence of the constant 0 in the term τ with the formula $\neg(X \to X)$, τ is transformed into a propositional formula α_τ. Therefore, there is a one-one correspondence between the propositional formulas and MV-terms. The following result can be proved by induction on the number of connectives in the formula α.

Proposition 1.5.3. *Let A be an MV-algebra, let α be a formula and we denote with $Var(\alpha) \subseteq \{X_{i_1}, \dots, X_{i_k}\}$ the set of all variables occurring in α, then:*

(i) For each A-valuation v, we have

$$v(\alpha) = \alpha^A(v(X_{i_1}), \dots, v(X_{i_k}))$$

where $\alpha^A\colon A^k \to A$ is the term function defined in definition 1.4.1.

(ii) A formula α is an A-tautology if and only if the MV-equation $\alpha = 1$ holds in A.

(iii) Two formulas α and β are semantically A-equivalent iff the equation $\alpha = \beta$ holds in A iff $\alpha^A = \beta^A$.

In *Lukasiewicz infinite-valued propositional calculus* one considers propositional formulas equipped with the relation of semantic [0, 1]-equivalence. Next result is a consequence of proposition 1.5.3 and it is an equivalent formulation of *Chang's Completeness Theorem* 1.3.5.

Proposition 1.5.4. *A formula α is a* [0,1]*-tautology iff, for every MV-algebra A, α is an A-tautology. For any two formulas α and β we have $\alpha^{[0,1]} = \beta^{[0,1]}$ iff $\alpha^A = \beta^A$ for all MV-algebras A.*

Since we are particularly interested in [0,1]-valuations, we are going to use a lighter notation calling *valuation* the [0,1]-valuation, *tautology* the [0,1]-tautology and with *semantic equivalence and consequence* the [0,1]-equivalence and [0,1]-consequence.

For each $\Theta \subseteq FORM$ the set of *semantic consequences* of Θ will be denoted with $\Theta^{\models}$. The set $\emptyset^{\models}$ will denote the set of all *semantic tautologies*, i.e. the set of all *valid* formulas.

Remark 1.5.5. Last results allow us to identify the term function $\alpha^{[0,1]}$ and the semantic equivalence classes of propositional formulas.

Definition 1.5.6. An axiom of the Łukasiewicz infinite-valued propositional calculus is a formula that can be written in any of the following way:

(A1) $\alpha \to (\beta \to \alpha)$

(A2) $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$

(A3) $((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha)$

(A4) $(\neg\alpha \to \neg\beta) \to (\beta \to \alpha)$

where α, β and γ are arbitrary formulas.

Definition 1.5.7. A proof from a set $\Theta \subseteq FORM$ is a finite string of formulas $\alpha_1, \ldots, \alpha_n$ with $n \geq 1$ such that for each $i \leq n$:

(i) α_i is an axiom, or

(ii) α_i belongs to Θ, or

(iii) there are $j, k \in \{1, \dots, i-1\}$ such that α_k coincides with the formula $(\alpha_j \to \alpha_i)$ (modus ponens)

The definition of proof allows us to give the definition of *provable formula*.

Definition 1.5.8. A formula α is provable from a set $\Theta \subseteq FORM$, in symbols $\Theta \vdash \alpha$, if there is a *proof* $\alpha_1, \dots, \alpha_n$ from Θ such that $\alpha_n = \alpha$.

By a *syntactic tautology* we shall mean a formula that is provable from the empty set. The set of provable formulas from Θ will be denoted with $\Theta^{\vdash}$. The set of *syntactic tautologies* will be denoted with $\emptyset^{\vdash}$.

Theorem 1.5.9. *The binary relation $\equiv$ on $FORM$ defined as follows*

$$p \equiv q \textit{ iff } \vdash p \leftrightarrow q$$

with $p, q \in FORM$, is an equivalence relation, called syntactic equivalence. *Moreover the relation $\equiv$ satisfies the following properties*

$$\textit{if } \alpha \equiv \gamma \textit{ and } \beta \equiv \delta \textit{ then } (\alpha \to \beta) \equiv (\gamma \to \delta)$$
$$\textit{if } \alpha \equiv \beta \textit{ then } \neg\alpha \equiv \neg\beta.$$

The equivalence class of a formula p will be denoted by $[p]$, i.e $[p] =_{def} \{q \in FORM \mid q \equiv p\}$.

Remark 1.5.10. Given a set $\Theta \subseteq FORM$ it is possible to defined another congruence $\equiv_\Theta$ in $FORM$ as follows

$$p \equiv_\Theta q \text{ iff } \Theta \vdash p \leftrightarrow q$$

If $\Theta = \emptyset$, the congruence $\equiv_\Theta$ coincide with the congruence $\equiv$. Moreover, if $p \equiv q$ then $p \equiv_\Theta q$ (see [8, 5.11]).

Theorem 1.5.11. *The quotient set $FORM/\equiv$ is an MV-algebra equipped with the operations $\neg$ and $\oplus$ and the constant 0, defined by:*

$$\neg[\alpha] =_{def} [\neg\alpha]$$

$$[\alpha] \oplus [\beta] =_{def} [\neg\alpha \to \beta]$$

$$0 =_{def} \neg[\emptyset^{\vdash}] = \{\alpha \in FORM \mid \text{ there is } \beta \in \emptyset^{\vdash} \text{ such that } \alpha \equiv \neg\beta\}$$

Remark 1.5.12. Given a set $\Theta \subseteq FORM$, the quotient $FORM/{\equiv_\Theta}$ is an MV-algebra with the operation $\neg$ and $\oplus$ and the constant 0 defined as in theorem 1.5.11.

The MV-algebra

$$L =_{def} \langle FORM/\equiv, 0, \neg, \oplus\rangle \tag{1.6}$$

is called the *Lindenbaum algebra of Łukasiewicz infinite-valued propositional calculus.*

Proposition 1.5.13. *Given the set $FORM$ with the relation $\equiv$ of syntactic equivalence. It follows that, for all p, $q \in FORM$:*

$$p \equiv q \text{ iff for any valuation } v,\ v(p) = v(q)$$

Proof.

$$\begin{aligned} p \equiv q &\text{ iff } v(p \to q) = 1 = v(q \to p) \text{ for all valuations } v \\ &\text{ iff } \min(1, 1 - v(p) + v(q)) = 1 = \min(1, 1 - v(q) + v(p)) \text{ for all } v \\ &\text{ iff } v(q) - v(p) \geq 0 \text{ and } v(p) - v(q) \geq 0 \text{ for all } v \\ &\text{ iff } v(p) = v(q) \text{ for all } v \end{aligned}$$

□

Proposition 1.5.14. *For every formulas p, p', $q \in FORM$:*

(i) $[\neg\neg p] = [p]$

(ii) $[p \to q] = \neg[p] \oplus [q]$

(iii) $[\neg p \to \neg q] = [q \to p]$

(iv) *if* $[p] = [p']$ *then* $[q \to p] = [q \to p']$

Proof. (i). It follows from theorem 1.5.11.

(ii). $\neg[p] \oplus [q] = [\neg p] \oplus [q] = [\neg\neg p \to q]$. It is sufficient to show that $[\neg\neg p \to q] = [p \to q]$. Let v be a valuation, then:

$$v(\neg\neg p \to q) = \min(1,\, 1 - v(\neg\neg p) + v(q)) = \min(1{,}1 - v(p) + v(q))$$

thus, the thesis follows from proposition 1.5.13.

(iii). $[\neg p \to \neg q] = [p] \oplus [q]$. On the other hand, $[q \to p] = [\neg q] \oplus [p]$.

(iv). For each valuation v:

$$\begin{aligned} v(q \to p) &= \min(1,\, 1 - v(q) + v(p)) \\ &= \min(1,\, 1 - v(q) - v(p')) = v(q \to p') \end{aligned}$$

The thesis follows again from proposition 1.5.13. □

Proposition 1.5.15. *For all $p, q \in FORM$ the following are equivalent:*

(i) $[p] \leq [q]$ in the MV-order on the Lindenbaum algebra;

(ii) $v(p) \leq v(q)$ for every valuation v;

(iii) $p \to q$ is valid.

Proof. $(iii) \leftrightarrow (ii)$.

$$\begin{aligned} p \to q \text{ is valid iff } & v(p \to q) = 1 \text{ for every } v \text{ valuation} \\ & \text{iff } 1 = \min(1,\, 1 - v(p) + v(q)) \text{ for all } v \\ & \text{iff } 0 \leq v(q) - v(p) \text{ for all } v \end{aligned}$$

$(iii) \leftrightarrow (i)$.

$$\begin{aligned} [p] \leq [q] & \text{ iff } \neg[p] \oplus [q] = 1 \\ & \text{iff } [p \to q] = [\emptyset^{\vdash}] \text{ by proposition 1.5.14} \\ & \text{iff } v(p \to q) = 1 \text{ for all } v. \\ & \text{iff } p \to q \text{ is valid.} \end{aligned}$$

□

Proposition 1.5.16. *For any $q_1, q_2, \dots, q_n \in FORM$:*

$$q_1 \to (q_2 \to (\cdots (q_n \to p)) \cdots) \text{ is valid iff } [q_1] \odot \cdots \odot [q_n] \leq [p] \tag{1.7}$$

Proof. See [8, 4.11] for details. □

1.6 Lindenbaum algebra and theories

In the section 1.5 we introduced the *Lindenbaum algebra* L of Ł$_\infty$. In this section we will give the notion of *theory* in Ł$_\infty$ and we will give some results involving quotients of L.

Proposition 1.6.1. *Up to isomorphism, the Lindenbaum algebra L coincides with the free MV-algebra over the generating set $\{[X_0], [X_1], \dots\}$ of logical equivalence classes of propositional variables.*

Proof. Let $\phi\colon L \to {Term_\omega}^{[0,1]}$ be the map defined by stipulating that $\phi([\alpha]) = \alpha^{[0,1]}$. The proposition 1.5.13 implies that ϕ is an isomorphism of the Lindenbaum algebra L onto the term algebra ${Term_\omega}^{[0,1]}$. In particular, the restriction of ϕ to the set of equivalence classes of propositional variables gives us a bijection from this set onto the set of projection functions $\{\pi_0, \pi_1, \dots\}$. The conclusion follows from proposition 1.4.6, thus $L \cong Free_\omega$. □

Definition 1.6.2. A *theory* of Łukasiewicz infinite-valued propositional calculus is a set Θ of formulas satisfying the following conditions:

(i) all axioms belong to Θ

(ii) if $\alpha \in \Theta$ and $(\alpha \to \beta) \in \Theta$, then $\beta \in \Theta$

Proposition 1.6.3. *For each set Θ of formulas:*

(i) $\Theta^\vdash$ *is the smallest theory containing* Θ.

(ii) Θ *is a theory iff* $\Theta = \Theta^\vdash$

(iii) if Θ *is a theory and* $\alpha \in \Theta$ *then* $\bigcup[\alpha] \in \Theta$

Proof. (i). From the definition 1.6.2 it follow that $\Theta^\vdash$ is a theory. In order to prove that $\Theta^\vdash$ is the smallest theory containing Θ, suppose that Γ is a theory such that $\Theta \subseteq \Gamma$. By

induction on n we shall prove that if $\alpha_1, \ldots, \alpha_n$ is a proof from Θ then $\alpha_n \in \Gamma$, thus $\Theta^{\vdash} \subseteq \Gamma$. If $n = 1$ then α_1 is either an axiom or a formula in Θ, in both cases it follows from definition 1.6.2 that $\alpha_1 \in \Gamma$. Suppose $n > 1$ and suppose that, for each proof from Θ, $\alpha_1, \ldots, \alpha_m$ with $m \leq n$, $\alpha_m \in \Gamma$. Suppose that $\alpha_1, \ldots, \alpha_n$ is a proof from Θ. If α_n is not an axiom and it does not belong to Θ, then there are $i, j \in \{1, \ldots, n\}$ such that α_j coincides with the formula $(\alpha_i \to \alpha_n)$ or, in other words α_n follows by modus pones from α_i and α_j. Since both $\alpha_1, \ldots, \alpha_i$ and $\alpha_1, \ldots, \alpha_j$ are proofs from Θ with $i, j \leq n$ it follows that $\alpha_i \in \Gamma$ and $(\alpha_i \to \alpha_j) \in \Gamma$. Thus, always by definition 1.6.2, we have that $\alpha_n \in \Gamma$.
(ii). It follows from (i).
(iii). If $\beta \in [\alpha]$ then $\alpha \to \beta \in \Theta^{\vdash}$ and, by the definition 1.6.2, $\beta \in \Theta$.

□

Definition 1.6.4. For each set $\Theta \subseteq FORM$ we define the set $\hat{\Theta}$ as follows:

$$p \in \hat{\Theta} \text{ iff } q_1 \to (q_2 \to \cdots (q_n \to p))) \cdots) \text{ is a tautology for some } q_1, \ldots, q_n \in \Theta \tag{1.8}$$

If $\Theta = \emptyset$ then $\hat{\Theta}$ is the set of all tautologies.

Proposition 1.6.5. *Given any two subsets Θ and Γ of FORM we have:*

(i) each valid sentence belongs to $\hat{\Theta}$;

(ii) $\Theta \subseteq \hat{\Theta}$;

(iii) $\hat{\hat{\Theta}} = \hat{\Theta}$;

(iv) $\Gamma \subseteq \Theta$ implies $\hat{\Gamma} \subseteq \hat{\Theta}$

Proof. (i). If $\Theta = \emptyset$, the conclusion is immediate and it follows from the definition. Suppose $\Theta \neq \emptyset$ and let $p \in \Theta$. By proposition 1.5.15, for every valid sentence $t \in FORM$ it follows: $p \to t$ is valid iff $[p] \leq [t]$ iff $[p] \leq 1$. Whence $p \to t$ is valid, hence $t \in \hat{\Theta}$.
(ii). Suppose $\Theta \neq \emptyset$. For every $p \in \Theta$ the sentence $p \to p$ is valid, then $p \in \hat{\Theta}$.
(iii). Assume $\Theta = \emptyset$. Then $p \in \hat{\hat{\emptyset}}$ iff $q_1 \to (q_2 \to \cdots (q_n \to p))) \cdots)$ is valid, with $q_1, \ldots, q_n \in \hat{\emptyset}$, iff $[q_1] \odot \cdots \odot [q_n] \leq [p]$, by proposition 1.5.16, iff $[p] \geq 1$ iff p is valid (proposition 1.5.15). Thus, $\hat{\hat{\emptyset}} = \hat{\emptyset}$. Suppose $\Theta \neq \emptyset$. In the light of (ii) it is sufficient to

show that $\hat{\hat{\Theta}} \subseteq \hat{\Theta}$. In order to prove this inclusion, first note that by (i), $\hat{\Theta} \neq \emptyset$. Whence by (ii), $\hat{\hat{\Theta}} \neq \emptyset$. If $p \in \hat{\hat{\Theta}}$ then the sentence $q_1 \to (q_2 \to \cdots (q_n \to p))) \cdots)$ is valid for suitable $q_1, \ldots, q_n \in \hat{\Theta}$. Then, by proposition 1.5.16, $[q_1] \odot \cdots \odot [q_n] \leq [p]$. For each $q_j \in \hat{\Theta}$, with $j = 1, \ldots, n$, there are $q_1^j, \ldots, q_{m(j)}^j \in \Theta$ such that $q_1^j \to (q_2^j \to \cdots (q_m(j)^j \to q_j))) \cdots)$ is valid. Hence, always by proposition 1.5.16, $[q_1^j] \odot \cdots \odot [q_{m(j)}^j] \leq [q_j]$. Using monotony of the operation $\odot$, it follows:

$$\prod_{j=1}^{n} \prod_{i=1}^{m(j)} [q_i^j] \leq \prod_{j=1}^{n} [q_j] \leq [p]$$

By another application of proposition 1.5.16, it follows that $p \in \hat{\Theta}$. $\square$

Remark 1.6.6. One can observe that the set $\hat{\Theta}$ is a theory.

Definition 1.6.7. For any $\Theta \subseteq FORM$, we denote with $[\Theta]$ the subset of L containing the equivalence classes of formulas in Θ, i.e. the set

$$[\Theta] = \{[p] \mid p \in \Theta\} \tag{1.9}$$

We denote with $F(\Theta)$ the filter generated by $[\Theta]$, i.e. the filter given by the intersection of all filters in L containing the set $[\Theta]$, and with $I(\Theta)$ the ideal defined as follows

$$I(\Theta) = F(\Theta)^* = \{[p] \in L \mid \neg[p] \in F(\Theta)\} = \{[p] \in L \mid [\neg p] \in F(\Theta)\}$$

Proposition 1.6.8. *For every $p \in FORM$ and $\Theta \subseteq FORM$ the following are equivalent*

(i) $[p] \in F(\Theta)$

(ii) $[p] \in F(\hat{\Theta})$

(iii) $p \in \hat{\Theta}$

Proof. If $\Theta = \emptyset$ then $\hat{\Theta}$ is the set of all valid sentences, $[\Theta] = \emptyset$ and $F(\Theta) = \{[\emptyset^{\vdash}]\} \subseteq L$. $F(\hat{\Theta})$ is the filter generated by the set of all $[p]$ such that p is valid, i.e. the filter generated by the element $[\emptyset^{\vdash}] \in L$. Therefore, $F(\Theta) = F(\hat{\Theta}) = \{[\emptyset^{\vdash}]\}$, and $[p] \in \{[\emptyset^{\vdash}]\}$ iff $[p] = [\emptyset^{\vdash}]$ iff p is valid iff $p \in \hat{\Theta}$.

Suppose $\Theta \neq \emptyset$.

(i)→(ii) it follows from proposition 1.6.5

(iii)$\to$(ii). Since $p \to p$ is a valid sentence, if $p \in \hat{\Theta}$ then $[p] \in F(\hat{\Theta})$.

(i)$\leftrightarrow$(iii).

$$\begin{aligned}[p] \in F(\Theta) &\text{ iff } [p] \text{ belongs to the filter generated by} \Theta/\equiv \\ &\text{ iff } [p] \geq y_1 \odot \cdots \odot y_n \text{ for suitable } y_i \in \Theta/\equiv \\ &\text{ iff } [p] \geq [q_1] \odot \cdots \odot [q_n] \text{ for suitable } q_i \in \Theta \\ &\text{ iff } q_1 \to (q_2 \to (\ldots (q_n \to p)) \ldots) \text{ is valid} \\ &\text{ iff } p \in \hat{\Theta}\end{aligned}$$

(ii)$\to$ (iii). $[p] \in F(\hat{\Theta})$ iff $[q_1] \odot \cdots \odot [q_n] \leq [p]$ for suitable $q_i \in \hat{\Theta}$ (as the previous point of the proof). It follows that, as in proposition 1.6.5, for all $i = 1, \ldots, n$ there exists $q^i_j \in \Theta$ such that $[q^i_1] \odot \cdots \odot [q^i_{m(i)}] \leq [q_i]$, then using monotony of multiplication

$$\prod_{i=1}^{n} \prod_{j=1}^{m(i)} [{q_j}^i] \leq \prod_{i}^{n} [q_i] \leq [p]$$

which shows that $p \in \hat{\Theta}$ (proposition 1.5.16). $\square$

Proposition 1.6.9. *For each filter F of L there is $\Theta \subseteq FORM$ such that $F(\Theta) = F$ and $\Theta = \hat{\Theta}$. In other words, for each filter F there is a theory Θ such that $F(\Theta) = F$.*

Proof. Define $\Theta = \{p \in FORM \mid [p] \in F\}$. It is easy to observe that $\Theta \neq \emptyset$ since $1 \in F$. In order to prove that $\Theta = \hat{\Theta}$, it is sufficient to check only the inclusion $\hat{\Theta} \subseteq \Theta$. For every $p \in FORM$:

$$\begin{aligned} p \in \hat{\Theta} &\implies [q_1] \odot \cdots \odot [q_n] \leq [p] \text{ with } q_i \in \Theta \\ &\implies y_1 \odot \cdots \odot y_n \leq [p] \text{ with } y_i \in F \\ &\implies y \leq [p] \text{ for some } y \in F \end{aligned}$$

Thus $[p] \in F$, hence $p \in \Theta$. $\square$

Corollary 1.6.10. *For each ideal I of L there is a theory $\Theta \subseteq FORM$ such that $I = I(\Theta)$.*

The quotient $L/I(\Theta)$ is called the *Lindenbaum algebra of* Θ and is denoted by $L(\Theta)$.

Theorem 1.6.11. *For every* $\Theta \subseteq FORM$

$$L/I(\Theta) \cong FORM/{\equiv_\Theta}$$

Proof. See [8, 5.13, 5.15] for details. □

Moreover, the following result holds for every MV-algebra and it is a consequence of proposition 1.6.1 and proposition 1.4.7.

Corollary 1.6.12. *For every countable* MV*-algebra* A *there is a* $\Theta \subseteq FORM$ *such that:*

$$A \cong L/I(\Theta)$$

Chapter 2

MV-algebras of McNaughton functions

In the previous chapter we introduced the notion of *free* MV-algebra over k-generators. Our main interest is the study of *ideals* of these particular MV-algebras, in particular, following [1], we shall give a characterization of *prime ideals* which will be an important tool in subsequent discussions concerning the problem of strong completeness. In order to pursue this aim we shall show some basic properties of the elements of $Free_k$ which depend on whether these elements are continuous [0,1]-valued functions on a compact Hausdorff topological space. In fact, from Chang's completeness theorem, each element of the free MV-algebra over k generators can be seen as piecewise linear continuous function with integer coefficients, defined over the cube $[0,1]^k$ with values in [0,1]. These functions are known as *McNaughton's functions*. The converse is given by *McNaughton's theorem* which states an isomorphism between the MV-algebra of McNaughton's functions over $[0,1]^k$ with values in [0,1] and the free MV-algebra over k-generators. Whence, McNaughton's functions stand to MV-algebras as {0,1}-valued functions stand to Boolean algebras.

2.1 McNaughton functions

Definition 2.1.1. A map $r\colon [0,1]^n \to [0,1]$ is a *McNaughton function* over the cube $[0,1]^n$ iff r is continuous and there is a finite number of linear polynomials $\rho_1 \ldots \rho_m$, called *linear constituents*, with integral coefficients such that, $\forall x \in [0,1]^n$ there is $j \in \{1, 2 \ldots, m\}$ with $r(x) = \rho_j(x)$.

The aforementioned can be generalized as follows:

Let k be an infinite cardinal, then a function $g\colon [0,1]^k \to [0,1]$ is a McNaughton function over $[0,1]^k$ iff there are ordinals $\alpha(0) < \cdots < \alpha(m-1)$, with $m \in \mathbb{N}$, and a McNaughton function f over $[0,1]^m$ such that for each $x \in [0,1]^k$

$$g(x) = f(x_{\alpha(0)}, \ldots, x_{\alpha(m-1)})$$

Remark 2.1.2. Since our main interest is the study of Łukasiewicz infinite-valued calculus with a denumerable set of propositional variables, we are going to study McNaughton functions defined over the Hilbert cube $[0,1]^\omega$. If we consider a McNaughton function $f : [0,1]^\omega \to [0,1]$, by definition 2.1.1, it depends on a finite number of variables. Let n be the maximum index of these variables. We can consider the initial segment $I_n = \{1, \ldots, n\}$, then the function f depends on a subset of the variables $x_0, \ldots, x_n$. Therefore, we can say that a map $f : [0,1]^\omega \to [0,1]$ is a McNaughton function over the Hilbert cube $[0,1]^\omega$ if and only if there is $n \in \{1, 2 \ldots\}$ and a McNaughton function r defined in $[0,1]^n$ such that

$$f(x) = r(x_0, \ldots , x_{n-1}) \;\; \forall x \in [0,1]^\omega$$

The set of McNaughton functions over $[0,1]^k$, with the following pointwise operations

$$\neg f(x) = 1 - f(x) \;\; \forall x \in [0,1]^k$$
$$(f \oplus g)(x) = \min(1, f(x) + g(x)) \;\; \forall x \in [0,1]^k$$

forms an MV-algebra. For each cardinal k, we denote with M_k the MV-algebra of McNaughton functions defined over $[0,1]^k$ and with M the MV-algebra of McNaughton functions defined over $[0,1]^\omega$.

The next proposition links McNaughton functions with the free MV-algebras introduced in the previous chapter.

Proposition 2.1.3. *For each cardinal k, if a function f belongs to a free MV-algebra $Free_k$ then f belongs to M_k.*

Proof. The projections and the function $\mathbf{0}$ which takes the value 0 over $[0,1]^k$ are McNaughton functions. If f and g are McNaughton function with linear constituents $\rho_1, \ldots, \rho_m$

and $\tau_1, \ldots, \tau_n$ then $f \oplus g$ is given by the linear polynomials $\rho_i + \tau_j$, for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$, together with the constant function $\mathbf{1}$. Then, McNaughton functions form a subalgebra of $[0,1]^{[0,1]^k}$. By lemma 1.4.3, all term functions are McNaughton functions. The conclusion follows from proposition 1.4.6. □

As we shall see, McNaughton's theorem gives us a characterization of elements of free MV-algebras, stating the converse of the previous proposition. The following result is simpler and it does not have the full strength of McNaughton's theorem, however it is useful for most applications. First of all, for each real-valued function f, we define

$$f^\wedge =_{def} (f \vee 0) \wedge 1$$

Lemma 2.1.4. *Let $g\colon [0,1]^n \to \mathbb{R}$ be a linear function with integer coefficient:*

$$g(x) = m_0 x_0 + \cdots + m_{n-1} x_{n-1} + m_n x_n + b$$

with $m_0, \ldots, m_n, b \in \mathbb{Z}$. Then $g^\wedge \in Free_n$

Proof. Let $m = |m_0| + |m_1| + \cdots + |m_{n-1}| + |m_n|$. The proof is by induction on m. If $m = 0$ then $g^\wedge$ coincides with the function $\mathbf{0}$ or the function $\mathbf{1}$. Whence it belongs to $Free_n$. Suppose that the lemma holds for $m - 1$. Without loss of generality, assume $|m_0| = \max(|m_0|, \ldots, |m_n|)$. If $m_0 > 0$, let $h = g - x_0$. Then we have

$$h = h(x_0, \ldots, x_n) = b + (m_0 - 1)x_0 + \cdots + m_n x_n$$

By induction hypothesis both $h^\wedge$ and $(h + 1)^\wedge$ belong to $Free_n$. We shall prove that for each $x = (x_0, \ldots, x_{n-1})$

$$(h + x_0)^\wedge = (h^\wedge \oplus x_0) \odot (h + 1)$$

It is clear that the identity holds whenever x is such that $h(x) > 1$ or $h(x) < -1$. If x is such that $h(x) \in [0,1]$, then $h^\wedge(x) = h(x)$ and $(h(x) + 1)^\wedge = 1$. Since $x_0 \in [0,1]$, $(h(x) + x_0)^\wedge = h(x) \oplus x_0$, then the equation holds. If $h(x) \in [-1,0]$ then $h^\wedge(x) = 0$ and

$(h(x)+1)^\wedge = h(x)+1$, the equation results from the identities

$$\begin{aligned}(h(x)+x_0)^\wedge &= \max(0, h(x)+x_0)\\ &= \max(0, x_0+h(x)+1-1)\\ &= x_0 \odot (h(x)+1)\end{aligned}$$

Thus the identity holds for each $x=(x_0,\dots,x_{n-1})$. By induction hypothesis and proposition 1.4.6, we have

$$(h+x_0)^\wedge = g^\wedge \in Free_n$$

If $m_0 < 0$ it is sufficient to apply the same argument to the function $1-g$ and show $(1-g)^\wedge \in Free_n$. Since $1-(1-g)^\wedge = g^\wedge$, we have $g^\wedge \in Free_n$. $\square$

Proposition 2.1.5. *For any two distinct points $x, y \in [0,1]^\omega$ there exists $f \in M$ such that $f(x)=0$ and $f(y)>0$.*

Proof. Let $x=(x_0,x_1,\dots)$ and $y=(y_0,y_1,\dots)$ be two distinct points of $[0,1]^\omega$. Without loss of generality, suppose $x_0<y_0$. Let r be a rational number such that $x_0<r<y_0$ and let $p(z)=az+b$ be a linear polynomial with integer coefficients such that $a>0$ and $r=-b/a$. By proposition 1.4.6 and lemma 2.1.4 it follows that the function $f(z)=p^\wedge(z)$ belongs to M, moreover, $f(x)=0$ and $f(y)>0$. $\square$

Theorem 2.1.6 (McNaughton's theorem)**.** *For each cardinal k, the free MV-algebra $Free_k$ is isomorphic to the set of McNaughton functions M_k.*

Proof. See [3] for details. $\square$

Remark 2.1.7. The free generating set of M_k is given by canonical projections. In M, we denote with $\{p_i \mid i=0,1\dots\}$ the set of canonical projections where $p_i\colon [0,1]^\omega \to [0,1]$ is given by

$$p_i(x) = x_i \;\forall x \in [0,1]^\omega$$

2.1.1 Simplexes, triangulations and indexes

In order to show some useful properties concerning McNaughton functions defined over $[0,1]^\omega$ and to give a geometrical investigation of prime ideals of finitely generated MV-algebras, in this section we will give some results which link McNaughton functions with the theory of convex polytopes.

We know that a d-dimensional simplex is a d-dimensional polytope with the least number of vertices. A point is a 0-dimensional simplex, a 1-dimensional simplex is a segment, a 2-dimensional simplex is a triangle and a 3-segment is a tetrahedron. Generally, a d-dimensional simplex has $d+1$ vertices. More formally, suppose that $u_0, \ldots, u_k$ are affinely independent points of $\mathbb{R}^k$, i.e. $u_1 - u_0, \ldots, u_k - u_0$ are linearly independent, then the simplex of vertices $u_0, \ldots, u_k$ is given by the set

$$\{\lambda_0 u_0 + \cdots + \lambda_k u_k \mid \sum_{i=1}^{k} \lambda_i = 1 \text{ and } \lambda_i \geq 0\ \forall i\}$$

Since our main interest is the study of McNaughton functions, we will consider simplexes which are contained in the cube $[0,1]^n$, with $n \in \mathbb{N}$. For each n-dimensional simplex $T \subseteq [0,1]^n$ we have:

(i) a list of vertices $v_0, v_1, \ldots, v_n$;

(ii) $d_i \geq 1$, the least common denominator of the coordinates of v_i, for each $i = 0, \ldots, n$;

(iii) a family of uniquely determined integers v_{ij} such that

$$v_i = (v_{i0}/d_i, \ldots, v_{i(n-1)}/d_i)$$
$$0 \leq v_{ij} \leq d_i \qquad (j = 0, \ldots, n-1)$$

with $gcd(v_{i0}, \ldots, v_{i(n-1)}) = 1$.

(iv) $v_i^{hom} \in \mathbb{Z}^{n+1}$, the homogeneous coordinates of the vertices of T

$$v_i^{hom} = (v_{i0}, \ldots, v_{i(n-1)}, d_i),\ i = 0, 1, \ldots, n$$

(v) the $(n+1) \times (n+1)$ matrix $\mathcal{M}_T$ whose ith row coincides with v_i^{hom}.

Definition 2.1.8. A triangulation $\mathcal{T}$ of $[0,1]^n$ is a set of n-dimensional simplexes such that the union of all simplexes in $\mathcal{T}$ coincides with $[0,1]^n$ and any two simplexes in $\mathcal{T}$ are either disjoint or intersect in a common face.

Definition 2.1.9. A simplex T is said to be *unimodular* if and only if $\det(\mathcal{M}_T) = 1$. A triangulation $\mathcal{T}$ of $[0,1]^n$ is said *unimodular* if and only if it is constituted by n-dimensional unimodular simplexes with rational vertices.

Definition 2.1.10. Let $\mathcal{T}$ be a unimoldular triangulation and let H be a rational hyperplane of $\mathbb{R}^n$, i.e. a set of points in $\mathbb{R}^n$

$$H = \{x \in \mathbb{R}^n \mid \sum_{i=1}^{n} m_i x_i = m_0\} \ (m_j \in \mathbb{Z}, \text{ for each } j = 0,\ldots,n)$$

where not all of $m_1,\ldots,m_n$ are zero. We say that the triangulation $\mathcal{T}$ *respects* the rational hyperplane H if each simplex of $\mathcal{T}$ is contained in H^+ or in H^-, where H^+ and H^- denote the two half-spaces defined by H.

Definition 2.1.11. Given a unimodular triangulation $\mathcal{T}$, a *refinement* of $\mathcal{T}$ is a unimodular triangulation $\mathcal{U}$ such that each simplex of $\mathcal{T}$ is the union of simplexes of $\mathcal{U}$.

Lemma 2.1.12. *Let $\mathcal{T}$ be a unimodular triangulation and H be a rational hyperplane of $\mathbb{R}^n$, then there exists a refinement $\mathcal{U}$ of $\mathcal{T}$ which respects H. Moreover, any two unimodular triangulations have a joint refinement that respects H.*

Notation and terminology. Given a set T we denote with $int(T)$ the interior of T and with $relint(T)$ the relative interior of T, namely the interior of T relative to the affine hull of T which is the smallest affine set containg T. The relative interior is more useful when we deal with low-dimensional sets placed in higher-dimensional spaces. Given a set T we denote with $conv(T)$ the convex hull of T, i.e. the smallest convex set containing T. The convex hull of a finite set T is given by all convex combinations of its elements.

Definition 2.1.13. For each $n \in \mathbb{N}$ and $0 \leq t \leq n$, a $(t+1)$-uple $\mathbf{u} = (u_0,\ldots,u_t)$ of vectors in $\mathbb{R}^n$ is called *index* if and only if $u_1,\ldots,u_t$ are linearly independent vectors and for some $\epsilon_1,\ldots,\epsilon_t \in \mathbb{R}^+$ the simplex

$$T = conv\{u_0, u_0+\epsilon_1 u_1, u_0+\epsilon_1 u_1+\epsilon_2 u_2,\ldots,u_0+\epsilon_1 u_1+\cdots+\epsilon_t u_t\} \qquad (2.1)$$

called **u**-simplex, is contained in $[0,1]^n$.

Remark 2.1.14. Given and index $\mathbf{u}$, for each $j = 0, \ldots, t$, we denote with u^j the j-uple $(u_0, u_1, \ldots, u_j)$. Since also u^j is an index, u^j-simplexes are well defined.

Proposition 2.1.15. *Let $\mathbf{u} = (u_0, \ldots, u_t)$ be an index. If T_1 and T_2 are $\mathbf{u}$-simplexes, then $T_1 \cap T_2$ contains some $\mathbf{u}$-simplex.*

Proof. By induction on t. Without loss of generality, suppose that $u_0 = 0$. It is easy to prove the cases $t = 0$ and $t = 1$. Suppose $t > 1$ and consider the **u**-simplexes

$$T_1' = conv\{0, \epsilon_1 u_1, \epsilon_1 u_1 + \epsilon_2 u_2, \ldots, \epsilon_1 u_1 + \cdots + \epsilon_{t-1} u_{t-1}\}$$

$$T_2' = conv\{0, \lambda_1 u_1, \lambda_1 u_1 + \lambda_2 u_2, \ldots, \lambda_1 u_1 + \cdots + \lambda_{t-1} u_{t-1}\}$$

$$T_1 = conv\{0, \epsilon_1 u_1, \epsilon_1 u_1 + \epsilon_2 u_2, \ldots, \epsilon_1 u_1 + \cdots + \epsilon_t u_t\}$$

$$T_2 = conv\{0, \lambda_1 u_1, \lambda_1 u_1 + \lambda_2 u_2, \ldots, \lambda_1 u_1 + \cdots + \lambda_t u_t\}$$

By induction hypothesis $T_1' \cap T_2'$ contains some u^{t-1}-simplex

$$T' = conv\{0, \alpha_1 u_1, \alpha_1 u_1 + \alpha_2 u_2, \ldots, \alpha_1 u_1 + \cdots + \alpha_t u_t\}$$

T_1 and T_2 are convex sets, therefore for each $x \in relint(T_1' \cap T_2')$ there are $\delta_1, \delta_2 > 0$ such that $x + \delta_1 u_t \in T_1$ and $x + \delta_2 u_t \in T_2$, whence, calling $\delta = \min\{\delta_1, \delta_2\}$, $x + \delta u_t \in T_1 \cap T_2$. The point $c = \frac{\alpha_1}{2} u_1 + \frac{\alpha_2}{2} u_2 + \cdots + \frac{\alpha_{t-1}}{2} u_{t-1}$ can be seen as a convex combination of vertices of T', then $c \in relint(T')$. Since $relint(T') \in relint(T_1' \cap T_2')$, there exists α such that $c + \alpha u_t \in T_1 \cap T_2$. Therefore,

$$T = conv\{0, \frac{\alpha_1}{2} u_1, \frac{\alpha_1}{2} u_1 + \frac{\alpha_2}{2} u_2, \ldots, c, c + \alpha u_t\}$$

is an **u**-simplex such that $T \subseteq T_1 \cap T_2$. □

Theorem 2.1.16. *Let $f\colon [0,1]^n \to [0,1]$ be a McNaughton function with linear constituents $\rho_1, \ldots, \rho_k$, there is a unimodular triangulation $\mathcal{T}$ of $[0,1]^n$ such that for each simplex $T \in \mathcal{T}$, f coincides with some ρ_j over T.*

Proof. See [3, 3.3.1., 9.1.2.] for details. □

The next lemma explains a standard tool to construct McNaughton functions starting from a triangulation and a {0,1}-valued map.

Lemma 2.1.17. *Let $\mathcal{T}$ be a unimodular triangulation and μ a {0,1}-valued map defined over the vertices of simplexes in $\mathcal{T}$. Let $f\colon [0,1]^n \to [0,1]$ be the unique function that is linear over each simplex of $\mathcal{T}$ and such that $\forall x$ vertex of a simplex of $\mathcal{T}$*

$$f(x) = \mu(x)$$

Then $f \in M_n$.

Proof. See [3, 9.1.4] for details. □

Proposition 2.1.18. *Let $f,g \in M$ then, for some n, there are McNaughton functions r and s over $[0,1]^n$ such that $f(z) = r(z_0, \ldots, z_{n-1})$ and $g(z) = s(z_0, \ldots, z_{n-1})$ for all $z \in [0,1]^\omega$. For each $\bar{x} = (x_0, \ldots, x_{n-1}) \in [0,1]^n$ there is a finite family $\Sigma = \{S_1, \ldots, S_h\}$ of n-dimensional simplexes in $[0,1]^n$ obeying these conditions:*

(i) *$\bar{x}$ is a common vertex of each simplex*

(ii) *$\exists 0 < \eta <\in \mathbb{R}$ such that $S_1 \cup S_2 \cup \cdots \cup S_h$ contains an open set W with $x \in W$ of the form:*

$$W = \left\{(y_0, \ldots, y_{n-1}) \in [0,1]^n \mid \left((y_0 - x_0)^2 + \cdots + (y_{n-1} - x_{n-1})^2\right)^{1/2} < \eta\right\}$$

(iii) *$\forall i = 1, \ldots, h$ there are linear polynomials ρ_i and σ_i with integer coefficients such that $r = \rho_i$ and $s = \sigma_i$ on S_i*

Proof. Let $f, g \in M$, then there are $n, t \in \mathbb{N}$ and $r \in M_n$ and $s \in M_t$ such that for each $z \in [0,1]^\omega$

$$f(z) = r(z_0, \ldots, z_q)$$

$$g(z) = s(z_0, \ldots, z_t)$$

With a similar argument of *remark* 2.1.2, assuming that $n = \max\{q, t\}$, we have that $f(z) = r(z_0, \ldots, z_n)$ and $g(z) = s(z_0, \ldots, z_n)$, $\forall z \in [0,1]^\omega$. Whenever we consider a McNaughton function f defined over $[0,1]^n$, we can find a unimodular triangulation $\mathcal{T}$ of $[0,1]^n$ such that f is linear over each simplex of $\mathcal{T}$ (theorem 2.1.16). Therefore, let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the two triangulations associated to r and s respectively. By lemma 2.1.12, we can find a joint refinement $\mathcal{V}$ of $\mathcal{T}_1$ and $\mathcal{T}_2$ such that both r and s are linear over each simplex of $\mathcal{V}$. We can obtain a new refinement $\mathcal{T}$ such that $\bar{x}$ is a vertex of some simplex of $\mathcal{T}$. In fact, since $\mathcal{T}$ is a triangulation of $[0,1]^n$, $\bar{x} \in \mathcal{V}$ for some simplex $T \in \mathcal{V}$, then we can refine the simplex T connecting each vertex of T with $\bar{x}$. Call Σ the set of all simplexes of $\mathcal{T}$ which have a vertex in $\bar{x}$, since $\mathcal{T}$ is a triangulation Σ is a finite set and there exists $0 < \eta \in \mathbb{R}$ such that the open ball centered in $\bar{x}$ of radius η is fully contained in the union of all simplexes of Σ. Hence (i) and (iii) hold. Finally, (iii) easily follows from our assumptions on $\mathcal{T}$. □

Definition 2.1.19. Given two functions $f, g \in M$ and $x \in [0,1]^\omega$, f and g have the same germ at x if and only if $f = g$ on some open set in $[0,1]^\omega$ containing x.

The following proposition emphasizes the role of direction derivatives of McNaughton functions.

Proposition 2.1.20. *Let $f \in M$, x, $y \in [0,1]^\omega$, $u = y - x$ then the one-side direction derivative at x*

$$f'(x;u) = \lim_{\lambda \to 0} \frac{f(x+\lambda u) - f(x)}{\lambda}$$

exists and is finite. Moreover, two functions f and g in M have the same germ at x if and only if $f(x) = g(x)$ and $f'(x; y-x) = g'(x; y-x)$ $\forall y \in [0,1]^\omega$.

Proof. These two properties are an immediate consequence if proposition 2.1.18. □

Proposition 2.1.21. *Let $f, g \in M$, by proposition 2.1.18 for some n there are two McNaughton function r and s defined over $[0,1]^n$ such that $f(z) = r(z_0, \ldots, z_n)$ and $g = s(z_0, \ldots, z_n)$ $\forall z \in [0,1]^\omega$, then we have that the two functions f and g have the same germ at x if and only if for each index $\mathbf{u} = (\bar{x}, u_1, \ldots, u_n)$, with $\bar{x} = (x_0, x_1, \ldots, x_n)$, the two functions r and s coincide over some $\mathbf{u}$-simplex.*

Proof. Suppose that f and g have the same germ at x. Therefore, there exists an open set A of $[0,1]^n$ such that $\bar{x} = (x_0, x_1, \ldots, x_n) \in A$ and the two functions r and s coincide over A. Let $\mathbf{u} = (\bar{x}, u_1, \ldots, u_n)$ be an index, then for some $\epsilon_1, \epsilon_2, \ldots, \epsilon_t \in \mathbb{R}$, such that the simplex $T = conv\{\bar{x}, \bar{x} + \epsilon_1 u_1, \ldots, \bar{x} + \epsilon_1 u_1 + \cdots + \epsilon_n u_n\}$ is contained in $[0,1]^n$. Therefore, we can choose another family of coefficients $\epsilon'_1, \epsilon'_2, \ldots, \epsilon'_n$ in a way that the $\mathbf{u}$-simplex $T' = conv\{\bar{x}, \bar{x} + \epsilon'_1 u_1, \ldots, \bar{x} + \epsilon'_1 u_1 + \cdots + \epsilon'_n u_n\}$ is smaller than T and it is fully contained in A. Whence, r and s coincide on the $\mathbf{u}$-simplex T'. Iterating this process for each index of the form $\mathbf{u} = (\bar{x}, u_1, \ldots, u_n)$, it follows that the two function r and s coincide over some $\mathbf{u}$-simplex for each index $\mathbf{u} = (\bar{x}, u_1, \ldots, u_n)$.
Conversely, suppose that the two functions r and s coincide over some $\mathbf{u}$-simplex for each index $\mathbf{u} = (\bar{x}, u_1, \ldots, u_n)$. Then we can consider the family Σ of $\mathbf{u}$-simplexes with arbitrary direction u_1 and fixed $u_2, \ldots, u_n$. Denoting with U the union of all these $\mathbf{u}$-simplexes, then the two functions r and s coincide over U. Whence, r and s coincide over the open set $int(U)$. In order to prove that $\bar{x} \in int(U)$, we can consider the open ball of center $\bar{x}$ and radius $\eta > 0$, denoted by $B_{\bar{x},\eta}$. If we consider $y \in B_{\bar{x},\eta}$ then $y = \bar{x} + h$ for some $h \in \mathbb{R}^n$. Therefore, there is an $\mathbf{u}$-simplex T in Σ such that $y \in T$. Whence, f and g coincide over an open set containing x. □

2.2 Ideals of M

Definition 2.2.1. For every non empty closed set $X \subseteq [0,1]^\omega$ it is possible to define two ideals of M as follows:

$$J_X = \{f \in M \mid f = 0 \text{ on } X\}$$
$$O_X = \{f \in M \mid f = 0 \text{ on some open set in } [0,1]^\omega \text{ containing } X\}$$

we write J_x and O_x instead of $J_{\{x\}}$ and $O_{\{x\}}$, respectively. Given an ideal J of M, it is possible to define the following subset of $[0,1]^\omega$

$$V_J = \{x \in [0,1]^\omega \mid J \subset J_x\} = \cap \left\{f^{-1}(0) \mid f \in J\right\}$$

Lemma 2.2.2. *Let A be a subalgebra of the MV-algebra M then for each $x \in A$ the ideal J_x is maximal in A.*

Proof. First of all, suppose that $A = M$. J_x is a proper ideal of A because the constant function $\mathbf{1}$ is not among its elements. If $f \in A \setminus J_x$ then $f(x) > 0$ and we can find an integer such that $nf(x) \leq 1$. It follows that

$$\neg nf = 1 - \underbrace{(f \oplus \cdots \oplus f)} \in J_x$$

Whence by proposition 1.1.19 J_x is maximal. To complete the proof it is sufficient to observe that given an MV-algebra B the intersection of a subalgebra and a maximal ideal of B is a maximal ideal in the subalgebra. □

Proposition 2.2.3. *The map*

$$J \mapsto V_J$$

is an inclusion reversing map from the set of ideals of M into the family of closed subsets of $[0,1]^\omega$. Moreover, $V_J \neq \emptyset$ for each proper ideal J of M.

Proof. The continuity of each $f \in M$ ensures that V_J is a closed subset of $[0,1]^\omega$, for each ideal J of M. It is easy to check that the map is inclusion reversing. Let J be a proper ideal of M. Suppose $V_J = \emptyset$ (absurdum hypothesis). The Hilbert cube $[0,1]^\omega$ is a compact Hausdorff space, then there are $f_1, \ldots, f_s \in J$ with $s \geq 1$ such that the intersection of their zerosets is empty. Let $f = f_1 \oplus \cdots \oplus f_s$, then $f \in J$ and the zeroset of f is empty. Since f attains minimum value > 0, there exists an integer $m \geq 1$ such that $mf(x) > 1$ for all $x \in [0,1]^\omega$. Thus, $\underbrace{f \oplus \cdots \oplus f}_{\text{m times}}$ takes value 1 for all $x \in [0,1]^\omega$. Since $f \in J$, it follows that $\mathbf{1} \in J$ and $J = M$, a contradiction. □

Theorem 2.2.4. *(i) The map $x \mapsto J_x$ is a one-one correspondence between the Hilbert cube $[0,1]^\omega$ and the set of maximal ideals of M.*

(ii) For each closed set $C \subseteq [0,1]^\omega$, $V_{J_C} = C$

(iii) For each proper ideal J in $[0,1]^\omega$, J_{V_J} is the intersection of all maximal ideals of M containing J.

Proof. (i). By lemma 2.2.2 the map $x \mapsto J_x$ is a one-one correspondence from $[0,1]^\omega$ into the set of all maximal ideals of M. In order to prove that the map is *onto* the set of all maximal ideal of M, let J be a maximal ideals of M. Since J is proper, by proposition 2.2.3, V_J is a nonempty closed set of $[0,1]^\omega$. Since for each $y \in V_J$, $J_y \supseteq J$, then V_J is a singleton, in particular $V_J = \{x\}$.

(ii). Trivially $C \subseteq V_{J_C}$. In order to prove the converse inclusion, consider $x \in [0,1]^\omega \setminus C$. By proposition 2.1.5, for each $y \in C$ there is $f_y \in M$ such that $f_y(y) = a_y > 0$ and $f(x) = 0$. By continuity, there is an open neighborhood U_y such that $f_y(z) > b_y = a_y/2$, for each $y \in C$ and $z \in U_y$. By the compactness of $[0,1]^\omega$ there is a finite family of functions $f_1, \dots, f_t \in M$ such that, taking $f = f_1 \oplus \dots \oplus f_t$, $f(x) = 0$ and $f(z) > \min(b_1, \dots, b_t) > 0$, for each $z \in C$. Then, for some integer $n \geq 1$, $\neg nf \in J_C$ and $\neg nf(x) = 1$, thus $x \notin V_{J_C}$. $\square$

Lemma 2.2.5. *O_X and J_X are respectively the smallest and the largest ideal J in M such that $V_J = X$.*

Proof. It is clear that $O_X \subset J_X$, so $V_{J_X} \subset V_{O_X}$ and $X \subset V_{J_X}$, by the definition of J_X. Let J be an ideal in M such that $V_J = X$. Then $J \subset J_X$. In fact, if we take an $f \in J$ by our hypothesis on J we have that $f = 0$ on X. We can also observe that $J \supset O_X$. We have only to prove that $V_{O_X} \subset X$. If $x \notin X$ then the open set $W = [0,1]^\omega \setminus X$ contains an open set U with $x \in U$ and $W \supset \bar{U}$, where $\bar{U}$ is the closure of U. It is possible to define a function $f \in M$ such that $f(x) = 0$ and $f(y) = 1$ $\forall y \notin U$, therefore, there is $g \in M$ such that $g(x) = 1$ and $g(y) = 0$ $\forall y \notin U$. In particular, $g = 0$ on the set $\bar{Y} = [0,1]^\omega \setminus \bar{U}$ that is a set containing X. Then, $g \in O_X$ and since $g(x) = 1$ we have that $x \notin V_{O_X}$. $\square$

We can observe that in the [0,1]-valued case there may be many ideals J in M with $V_J = X$, in contrast with the two-valued case. The following result gives us a characterization for the uniqueness of J such that $V_J = X$.

Theorem 2.2.6. *For each $x = (x_0, x_1, \dots) \in [0,1]^\omega$ the following are equivalent:*

(i) There is only one ideal J in M with $V_J = \{x\}$

(ii) The set $\{1, x_0, x_1 \dots\}$ is linearly independent in the vector space $\mathbb{R}$ over $\mathbb{Q}$.

Proof. *(i)* $\to$ *(ii)*. Assume the negation of *(ii)* saying that

$$0 = a + b_0x_0 + b_1x_1 + \dots + b_nx_n \tag{2.2}$$

for some nonzero (n+2)-uple $(a, b_0, \dots, b_n) \in \mathbb{Q}^{n+2}$. Without loss of generality we can suppose $a, b_0, \dots, b_n \in \mathbb{Z}$. Consider the following function:

$$f(z) = a + b_0 z_0 + \cdots + b_n z_n \quad \text{for all } z \in [0,1]^\omega \tag{2.3}$$

We can observe that f is not necessarily in M, therefore we can consider $g = (f \vee -f) \wedge 1$. Then $g \in M$ and there is no open set in $[0,1]^\omega$ containing x such that $g = 0$ in this open set. For otherwise, by 2.1.20, we have $g'(x; y - x) = 0$ for all $y \in [0,1]^\omega$. In particular, taking the direction derivative along the coordinate axis, we obtain from (2.3): $b_0 = b_1 = \cdots = b_n = 0$. By (2.2) we have $a = 0$, thus we have a contradiction with our hypothesis on $(a, b_0, b_1, \dots, b_n)$. Then we have that $g \in J_x \setminus O_x$ and, by lemma 2.2.5, $V_{J_x} = V_{O_x} = \{x\}$. Therefore *(i)* does not hold.

In order to prove $(ii) \leftarrow (i)$, assume the negation of (i), thus by lemma 2.2.5 there is $g \in J_x \setminus O_x$. By Proposition 2.1.20 there is $y \in [0,1]^\omega$ such that

$$g'(x; y - x) \neq 0 \tag{2.4}$$

and by Proposition 2.1.18 there is a McNaughton function $s : [0,1]^n \to \mathbb{R}$ with:

$$g(z) = s(z_0, \dots, z_{n-1})\text{for all } z \in [0,1]^\omega \tag{2.5}$$

and there is $0 < \epsilon \in \mathbb{R}$ such that for some integers $a, b_0, \dots, b_{n-1}$ such that

$$s(z_0, \dots, z_n) = a + b_0 z_0 + \cdots + b_{n-1} z_{n-1} \tag{2.6}$$

for all points $(z_0, \dots, z_n)$ in the segment joining $\bar{x} = (x_0, \dots, x_{n-1})$ and $\bar{x} + \epsilon\bar{u}$ where $\bar{u} = (x_0 - y_0, \dots, x_{n-1} - y_{n-1})$. Since $g \in J_x$ from (2.5) and (2.6) we have:

$$0 = s(\bar{x}) = a + b_0 x_0 + \cdots + b_n x_n \tag{2.7}$$

If $a = b_0 = \cdots = b_n = 0$ by (2.6) we have $s'(\bar{x}; \bar{u}) = 0$ and hence $g'(x, y - x) = 0$, thus contradicting (2.4). Therefore, by (2.7) the set $\{1, x_0, \dots, x_{n-1}\}$ is not linearly independent in $\mathbb{R}$ as a $\mathbb{Q}$-vectorspace, thus *(ii)* does not hold. □

For every nonempty subset X of $[0,1]^k$ the map

$$\rho\colon f \in M_k \to f|_X \in M_k|_X$$

is a surjective homomorphism. Suppose $X = V_J$ for some proper ideal J of A, then it follows that $Ker(\rho) = J_{V_J}$. By lemma 1.1.22 we have the following result.

Proposition 2.2.7. *For each $J \in \mathcal{I}(M_k)$ the map*

$$f/J \mapsto f|_{V_J}$$

is an isomorphism from M_k/J onto $M_k|_{V_J}$ if and only if J is an intersection of maximal ideal of M_k.

Theorem 2.2.8. *Each proper principal ideal of M_k is an intersection of maximal ideals.*

Proof. See [3, 3.4.9] for details. □

Lemma 2.2.9. *Given $f, g \in M_k$, we have that*

$$g \in \langle f \rangle \text{ iff } g^{-1}(0) \supset f^{-1}(0)$$

Proof. See [3, 3.4.8] for details. □

Lemma 2.2.10. *Let $f, g \in M$, $x \in [0,1]^\omega$ and $f(x) = g(x) = 0$. If for every $y \in [0,1]^\omega$, $f'(x; y - x) = 0$ implies $g'(x; y - x) = 0$ then for some $m \in \omega$ and open set U containg x we have:*

$$\underbrace{f \oplus \cdots \oplus f}_{m\, times} \geq g$$

Proof. Consider n, r, s, $\Sigma = \{S_1, S_2 \ldots S_h\}$ as in the proposition 2.1.18.
Let $\bar{x} = (x_0, x_1, \ldots, x_{n-1})$ *and* $e_i{}^j$ be the j^{th} extremal point of S_i, for each $i = 1, \ldots, h$ and $j = 1, \ldots, n+1$. Let $u_i{}^j = e_i{}^j - \bar{x}$, since $r(\bar{x}) = s(\bar{x}) = 0$ then, by our hypotheses, there exists n_i such that

$$n_i r'(\bar{x}; u_i{}^j) \geq s(\bar{x}; u_i{}^j) \quad \text{for all}\, j = 1, \ldots, n+1$$

Again by proposition 2.1.18, on the simplex S_i the function $n_i r - s$ has nonnegative directional derivative at $\bar{x}$ along each direction $u_i{}^0, \ldots, u_i{}^{n+1}$. Whence by linearity, $n_i r \geq s$ on S_i. Letting $m = max(n_1, \ldots, n_h)$, we have $mr \geq s$ on the set $\bigcup_i S_i$, and hence, still by Proposition 2.1.18, $mr \geq s$ on some open set in $[0,1]^\omega$ containing $\bar{x}$. By the basic properties of the Hilbert cube we have that $mf \geq g$ on some open set $U \subset [0,1]^\omega$ containing x. Since $g \leq 1$ then we have

$$\underbrace{f \oplus \cdots \oplus f}_{m\, times} = 1 \wedge mf \geq g \quad on\ U$$

□

Definition 2.2.11. Given $Y \subseteq [0,1]^\omega$, denoting with $Fr(Y)$ the boundary of Y, the function

$$\delta : Fr(Y) \to \mathcal{P}([0,1]^\omega)$$

is a (principal) germination of Y if and only if δ assigns to each $x \in Fr(Y)$ a (principal) proper filter $\delta(x)$ over the power set of $[0,1]^\omega$.

Definition 2.2.12. Given $Y \subset [0,1]^\omega$ and a germination δ of Y. We define the set $J_{Y,\delta}$ as follows:

$$f \in J_{Y,\delta} \quad \text{iff} \quad f = 0 \text{ on } Y \text{ and}, \forall x \in Fr(Y),\ \{y \in [0,1]^\omega \mid f'(x; y - x) = 0\} \in \delta(x) \quad (2.8)$$

Theorem 2.2.13. *(i) Let $X \subset [0,1]^\omega$ be a nonempty closed set and δ a germination of X. Then $J_{X,\delta}$ is a proper ideal in M and $V_{J_{X,\delta}} = X$.*

(ii) Let J be a proper ideal in L and $V_J = X$. Then $J = J_{X,\delta}$ for some germination δ of X.

Proof. *(i)* Denote $J_{X,\delta} = J$ by proposition 2.1.20. It is easy to see that J is an ideal in M. It is also clear that $1 \notin J$ because $X \neq \emptyset$. In other to prove that $V_J \subset X$, by lemma 2.2.5 it is sufficient to prove $O_X \subset J$. Consider $f \in O_X$ then $f = 0$ on X and for each $x \in X \supset Fr(X)$ we have $f = 0$ on some open set containing x. Therefore, by proposition 2.1.20 the set $\{y \in [0,1]^\omega \mid f'(x; y - x) = 0\}$ coincides with $[0,1]^\omega$ that is an element of $\delta(x)$. Thus $f \in J$.

(ii) To avoid trivialities assume $J \neq \{0\}$. In order to define a germination δ, for each $x \in Fr(X)$, we can consider the family of subsets $\delta(x)$ in $[0,1]^\omega$ defined as follow:

$$Y \in \delta(x) \qquad \text{iff} \qquad Y \supset \{y \in [0,1]^\omega \mid f'(x; y - x) = 0\} \text{ for some } f \in J \qquad (2.9)$$

We can observe that $\mathbf{0} \in J$, then $[0,1]^\omega \in \delta(x)$. Since $f'(x; y-x) = 0$ for all $f \in J$, then x is a common element of all $Y \in \delta(x)$, thus $\emptyset \notin \delta(x)$. It is easy to see that if $Y_1 \in \delta(x)$ and $Y_1 \subset Y_2 \subset [0,1]^\omega$ then $Y_2 \in \delta(x)$. If Y_1, $Y_2 \in \delta(x)$, then there are f_1, $f_2 \in J$ such that $\{y \in [0,1]^\omega \mid f_i(x; y-x) = 0\} \subset Y_i$ for $i = 1,2$. Letting $g = f_1 \oplus f_2$, then $g \in L$ and $g = f_1 + f_2$ on some open set containing x because $f_1(x) = f_2(x) = 0$ as $x \in Fr(X) \subset V_J = X$. It follows that $\{y \in [0,1]^\omega \mid g'(x; y-x) = 0\} \subset Y_1 \cap Y_2$. Therefore $\delta(x)$ is a germination of X. Now we shall show that $J = J_{X,\delta}$. By the eq. (2.9) and the definition of $J_{X,\delta}$ it is clear that $J \subset J_{X,\delta}$. In other to prove the other inclusion, consider $g \in J_{X,\delta}$. For each $x \in Fr(X)$ there is a function $f_x \in J$ such that $\{y \in [0,1]^\omega \mid g'(x; y-x) = 0\} \supset \{y \in [0,1]^\omega \mid {f_x}'(x; y-x) = 0\}$. Then by lemma 2.2.10 there is an open set U_x containing x and $m_x \in \omega$ such that

$$g \leq \underbrace{f_x \oplus \cdots \oplus f_x}_{m_x \in \omega} = \hat{f}_x \qquad \text{on } U_x \qquad \text{and } \hat{f}_x \in J \tag{2.10}$$

The family $\{U_x \mid x \in Fr(X)\}$ is an open cover of the close set $Fr(X)$. Then by compactness there are $x_1, \ldots, x_k \in Fr(X)$ such that the set $W = U_{x_1} \cup \cdots \cup U_{x_k}$ still covers $Fr(X)$. Let $\hat{f} = \hat{f}_{x_1} \oplus \cdots \oplus \hat{f}_{x_k}$ then from 2.10 we have $\hat{f} \in J$ and $\hat{f} \geq g$ on W. Since $g = 0$ on X, then

$$g \leq \hat{f} \qquad \text{on some open set } U \text{ containing } x \tag{2.11}$$

Let $b \in M$ defined by $b = g \vee \hat{f}$, then by 2.11 $b = \hat{f}$ on U whence $\hat{f}^* \cdot b = 0$ on U. Since $U \supset X = V_J$ it follows that $\hat{f}^* \cdot b \in J$. Since $\hat{f} \in J$ and $\hat{f}^* \cdot b \in J$ we obtain $b = \hat{f}^* \cdot b \oplus \hat{f} \in J$. Finally, $g \leq b$ and, therefore, $g \in J$. □

2.3 Simple and semisimple MV-algebras

Definition 2.3.1. An MV-algebra A is called *simple* iff it has exactly two ideals. In other words, an MV-algebra A is simple if A is nontrivial and $\{0\}$ is its only proper ideal.

Theorem 2.3.2. *For every MV-algebra A the following conditions are equivalent:*

(i) A is simple

(ii) A is nontrivial and for every nonzero element $x \in A$ there is no integer $n > 0$ such that $1 = \underbrace{x \oplus \cdots \oplus x}_{n \text{ times}}$

(iii) A is isomorphic to a subalgebra of $[0,1]$

Proof. $(i) \leftrightarrow (ii)$. Suppose that A is simple then the ideal $\{0\}$ is maximal in A, then (ii) follows from proposition 1.1.19. Conversely, (ii) states that $\{0\}$ is a maximal ideal of A, hence A is simple.

$(iii) \rightarrow (ii)$. It is clear that (ii) is satisfied by all subalgebras of $[0,1]$.

$(ii) \rightarrow (iii)$. Assume A is simple. If A has cardinality k, then A is isomorphic to the quotient $Free_k/J$ (by proposition 1.4.7). Since A is simple, the ideal J must be maximal in $Free_k$ (proposition 1.1.25). Therefore, by theorem 2.2.4, there exists a uniquely determined point $x \in [0,1]^k$ such that $J = J_x$. Therefore, J coincides with the intersection of all maximal ideals of $Free_k$ containing J. Applying proposition 2.2.7, it follows that A is isomorphic to the MV-algebra $Free_k|_{\{x\}} = \pi_x(Free_k)$, where $\pi_x : Free_k \rightarrow [0,1]$ is the map given by $\pi_x(f) = f(x)$. Whence A is isomorphic to a subalgebra of $[0,1]$. □

Let A be an MV-algebra, we denote with $Rad(A)$ the radical of A, i.e. the intersection of all maximal ideals of A.

Definition 2.3.3. An MV-algebra A is said to be *semisimple* iff A is nontrivial and $Rad(A) = \{0\}$.

It is clear that every simple MV-algebra is semisimple. We can also observe that, in the light of proposition 1.1.25, given an ideal J of an MV-algabra A the quotient A/J is simple if and only if J is maximal. Therefore by theorem 2.3.2 the quotient A/J is isomorphic to a subalgebra of $[0,1]$ if and only if J is maximal. As an immediate consequence of Birkhoff's theorem we have the following result.

Proposition 2.3.4. *An MV-algebra A is semisimple if and only if it is a subdirect product of subalgebras of* $[0,1]$.

Remark 2.3.5. From theorem 2.3.2 it follows that an MV-algebra A is semisimple if and only if A is a subdirect product of simple MV-algebras.

Corollary 2.3.6. *Every free MV-algebra is semisimple.*

Proof. It follows from proposition 1.4.6 □

Lemma 2.3.7. *Given an MV-algebra A and an ideal J of A, the quotient algebra A/J is semisimple if and only if J is an intersection of maximal ideals of A.*

Proof. Suppose that A/J is semisimple, if $\{M_i\}_{i\in I}$ denotes the family of all maximal ideals of A/J and h_J denotes the natural projection, then

$$J = h_J^{-1}(\{0\}) = h_J^{-1}(\bigcap_{i\in I} M_i) = \bigcap_{i\in I} h_J^{-1}(M_i)$$

By proposition 1.1.25, J is an intersection of maximal ideals of A. Conversely, suppose that J is an intersection of maximal ideals of A, then J is the intersection of all maximal ideals of containing J. Let $\{M_i\}_{i\in I}$ denote this family, again by proposition 1.1.25, the set $\{h_J(M_i)\}_{i\in I}$ denotes the family of all maximal ideals of A/J and $h_J(J) = Rad(A/J)$. Whence A/J is semisimple. □

Theorem 2.3.8. *An MV-algebra A with k many generators is semisimple if and only if for some nonempty closed set $X \subseteq [0,1]^k$, A is isomorphic to the MV-algebra of restrictions to X of all functions in $Free_k$*

Proof. Suppose that A is semisimple. By proposition 1.4.7, there exists and ideal J of $Free_k$ such that $A \cong Free_k/J$. By proposition 2.2.7 and lemma 2.3.7, A is isomorphic to the MV-algebra of restrictions to V_J of functions of $Free_k$. The converse direction is a consequence of proposition 2.3.4. □

Following [4], we can give the following definition.

Definition 2.3.9. An MV-algebra A is *strongly semisimple* is all its principal quotients are semisimple.

Remark 2.3.10. Given any MV-algebra A, since $\{0\}$ is a principal ideal of A, every strongly semisimple MV-algebra is semisimple.

The following results, known as Wójcicki's theorem, follows from theorem 2.2.8 and lemma 2.3.7.

Theorem 2.3.11. *Given an MV-algebra A such that $A \cong Free_k/J$ with J principal ideal of $Free_k$, then A is semisimple.*

2.4 Quotients of M

Theorem 2.4.1. *Let J be a proper ideal in M, and $X = V_J$, then the following conditions are equivalent:*

(i) J is the set of functions vanishing on X, namely $J = J_X$

(ii) M/J is the MV-algebra of restrictions to X of functions in M

(iii) $J = J_{X,\delta^\bullet}$ where $\delta^\bullet$ is the map that assigns to each $x \in Fr(X)$ the principal ultrafilter $\delta^\bullet(x) = \{Y \supset [0,1]^\omega \mid x \in Y\}$

(iv) J is the intersection of all maximal ideals in M containing J

(v) M/J is isomorphic to a subalgebra of a direct product of simple MV-algebras

Proof. (i) $\to$ (ii). Consider f/J, $g/J \in L/J$, then:

$$
\begin{aligned}
f/J = g/J &\text{ iff } \neg f \odot g \oplus f \odot \neg g \in J \\
&\text{ iff } \neg f \odot g \oplus f \odot \neg g = 0 \text{ on } X \\
&\text{ iff } \neg f \odot g = 0 \text{ and } f \odot \neg g = 0 \text{ on } X \\
&\text{ iff } f = g \text{ on } X
\end{aligned}
$$

Then the map $i\colon f/J \to f\big|_X$ induces a bijection of M/J onto the MV-algebra of restrictions to X of the functions in M. The map is also an isomorshism.

(ii) $\leftrightarrow$ (v). It follows from theorem 2.3.8.

(i) $\leftrightarrow$ (iii). In order to show that $J = J_X = J_{X,\delta^\bullet}$, we can observe that

$$
\begin{aligned}
f \in J_{X,\delta^\bullet} &\text{ iff } f = 0 \text{ on } X \text{ and, for all } x \in Fr(X),\ \{y \in [0,1]^\omega \mid f'(x; y - x) = 0\} \in \delta^\bullet(x) \\
&\text{ iff } f = 0 \text{ on } X \text{ and } f'(x; 0) = 0 \\
&\text{ iff } f = 0 \text{ on } X \\
&\text{ iff } f \in J_X
\end{aligned}
$$

(iv) $\leftrightarrow$ (v). By lemma 2.3.7.

(iv) $\to$ (i). Suppose that J is the intersection of all maximal ideals of M containing J, i.e.

$$J = \bigcap_{I \in \mathcal{M}(M)} I \qquad \text{such that } I \supset J$$

By theorem 2.2.4, for each $I \in \mathcal{M}(M)$ there exists $y \in [0,1]^\omega$ such that $I = J_y$. It is clear that $J_X \subset J_x$ for all $x \in X$. If $f \in J$ then $f(x) = 0$ for all $x \in X = V_J$, hence $f \in J_X$. Conversely, if $f \in J_X$, then $f \in J_x$ $\forall x \in X = V_J$. Therefore,

$$f \in \bigcap_{x \in X} J_x \subseteq \bigcap_{J_x \subseteq J} J_x = J$$

hence $f \in J$. □

Theorem 2.4.2. *A proper ideal J of M has the equivalent properties shown in theorem 2.4.1 if J satisfies at least one of the following conditions:*

(i) J is maximal;

(ii) J is finitely generated;

(iii) J is generated by McNaughton functions corresponding to the negations of the axioms for MV_n algebras with $n \geq 2$;

(iv) for each $x \in Fr(V_J)$ the set $\{1, x_0, x_1 \ldots\}$ is linearly independent in the vector space $\mathbb{R}$ over $\mathbb{Q}$;

Proof. *(i)*. If J is maximal then it is clear that it satisfies (iv) of theorem 2.4.1.
(ii). Assume J is generated by one element $f \in M$ and denote the zeroset of f with $X = f^{-1}(0)$. Our purpose is to prove that $J = J_X$. It is easy to check that $J \subset J_X$. Assume $g \in J_X$ and let $x \in Fr(X)$. Let $y \in [0,1]^\omega$ be such that $g'(x; y - x) \neq 0$. Let $u = (y - x)$. Then for all sufficiently small $\epsilon \geq 0$ the point $x_\epsilon = x + \epsilon u$ is not in X, because $g = 0$ at $x \in X$ and g is linear on the segment $[x, x_\epsilon]$ by proposition 2.1.18. From $x_\epsilon \notin X$, $f(x_\epsilon) \neq 0$ we have that $f'(x; y-x) \neq 0$, again by proposition 2.1.18. Then by lemma 2.2.10 there is $m_x \in \omega$ and an open set U_x containing x such that $\underbrace{f \oplus \ldots f \oplus}_{m_x \text{ times}} \geq g$ on U_x. As the final part of theorem 2.2.13 in *(ii)* and recalling J is generated by f, we finally get $g \in J$.
(iii). By [6] together with theorem 2.4.1.
(iv). Let $X = V_J$. Then by lemma 2.2.5 it is sufficient to prove $O_X = J_X$, for then $J = J_X$. Assume $f \in J_X \setminus O_X$ (absurdum hypothesis), let $Z = f^{-1}(0)$ and let $int(Z)$ denote the interior of Z. From $f \in J_X$ we get $Z \supset X$; from $f \notin O_X$ we get $X \not\subset int(Z)$. Then there is

$x \in X \setminus int(Z)$ and $x \in X \setminus int(X)$. Whence $x \in Fr(X)$, as X is closed. Since $x \notin int(Z)$ then $f \notin O_x$. Thus $f \in J_x \setminus O_x$. By theorem 2.2.6 the set $\{1, x_0, x_1, \dots\}$ is not linearly independent in $\mathbb{R}$ seen as a $\mathbb{Q}$-vector space. This contradicts our assumption. □

2.5 Prime ideals of M_n

Proposition 2.5.1. *Given an index* $\mathbf{u} = (u_0, \dots, u_t)$*, with* $0 \le t \le n$*, the subset* $J_{\mathbf{u}}$ *of* $Free_n$*, defined as follows*

$$f \in J_{\mathbf{u}} \textit{ iff } f^{-1}(0) \textit{ contains some } \mathbf{u}\textit{-simplex}$$

is an ideal of M_n*.*

Proof. In order to prove that $J_{\mathbf{u}}$ is closed under minorants, suppose that $f \in J_{\mathbf{u}}$ and consider $g \in M_n$ such that $g \le f$. Trivially, $f^{-1}(0) \subseteq g^{-1}(0)$, whence $g^{-1}(0)$ contains some $\mathbf{u}$-simplex, hence $g \in J_{\mathbf{u}}$. If $f, g \in J_{\mathbf{u}}$ then by definition there are two $\mathbf{u}$-simplexes T_1 and T_2 such that

$$f^{-1}(0) \supseteq T_1 \qquad \text{and} \qquad g^{-1}(0) \supseteq T_2$$

By proposition 2.1.15 it follows that there exists a $\mathbf{u}$-simplex T which is contained in $T_1 \cap T_2$. Let us consider the zeroset of $f \oplus g$ then $(f \oplus g)^{-1}(0) \supseteq f^{-1}(0) \cap g^{-1}(0)$, hence $f \oplus g \in J_{\mathbf{u}}$. □

Our next aim is to show that $J_{\mathbf{u}}$ is a prime ideal of M_n. Moreover, we shall see that every prime ideal J of M_n has the form $J = J_{\mathbf{u}}$ for some index $\mathbf{u}$.

Definition 2.5.2. Given a unimodular triangulation $\mathcal{T}$ of $[0,1]^n$ and an index $\mathbf{u} = (u_0, \dots, u_t)$, with $0 \le t \le n$, we define the set

$$\mathcal{T}^{\mathbf{u}} = \bigcap \{F \mid F \text{ is a simplex of } \mathcal{T} \text{ and } F \text{ contains some } \mathbf{u}\text{-simplex}\}$$

which is still a simplex of $\mathcal{T}$ containing some $\mathbf{u}$-simplex (by proposition 2.1.15). Recalling the notation u^j, it follows that T^{u^j} is well defined for each $j = 0, \dots, t$.

Remark 2.5.3. From now on, all triangulations we are going to consider will be unimodular.

Proposition 2.5.4. *For any index* $\mathbf{u} = (u_0, \ldots, u_t)$*, with* $0 \le t \le n$*, the ideal* $J_{\mathbf{u}}$ *is a prime ideal of* M_n*.*

Proof. In order to see that $J_{\mathbf{u}}$ is prime, suppose that $f \notin J_{\mathbf{u}}$ and $g \notin J_{\mathbf{u}}$. Applying theorem 2.1.16 together with lemma 2.1.12, there exists a unimodular triangulation $\mathcal{T}$ such that f, g, $f \wedge g$ are linear over each simplex of $\mathcal{T}$. If f vanishes over $\mathcal{T}^{\mathbf{u}}$ (absurdum hypothesis), then f vanishes over some $\mathbf{u}$-simplex $T \subseteq \mathcal{T}^{\mathbf{u}}$. Whence $f \in J_{\mathbf{u}}$, a contradiction. It follows that $f(x) > 0$ for some $x \in \mathcal{T}^{\mathbf{u}}$. Similarly, $g(y) > 0$ for some $y \in \mathcal{T}^{\mathbf{u}}$. From the assumption about $\mathcal{T}$ it follows that f and g are positive over $relint(\mathcal{T}^{\mathbf{u}})$ and $f \wedge g$ is linear over $\mathcal{T}^{\mathbf{u}}$. Therefore, we have that $f \le g$ or $g \le f$ over $\mathcal{T}^{\mathbf{u}}$, in either case $f \wedge g \neq 0$. Thus, $J_{\mathbf{u}}$ is prime. □

Definition 2.5.5. Given an index $\mathbf{u} = (u_0, \ldots, u_t)$, we define the set

$$\zeta(u^0) = \bigcap \{H \mid u_0 \in H\} \tag{2.12}$$

and for each $i = 1, \ldots, t$

$$\zeta(u^j) = \bigcap \{H \mid conv\{u_0, u_0 + \epsilon_1 u_1, \ldots, u_0 + \epsilon_1 u_1 + \cdots \epsilon_j u_j\} \subseteq H\} \tag{2.13}$$

or equivalently,

$$\zeta(u^j) = \bigcap \{H \mid T \subseteq H \text{ for some } u^j\text{-simplex } T\} \tag{2.14}$$

A translation of $-u_0$ of $\zeta(u^j)$ (for each $0 \le i \le t$) yields its associated linear space

$$\lambda(u^j) = \zeta(u^j) - u_0 = \{x \in \mathbb{R}^n \mid (x + u_0) \in \zeta(u^j)\} \tag{2.15}$$

Notation. We shall write $\zeta(\mathbf{u})$ instead of $\zeta(u^t)$ and $\lambda(\mathbf{u})$ instead of $\lambda(u^t)$.

Remark 2.5.6. Given a triangulation $\mathcal{T}$ and an index $\mathbf{u} = (u_0, \ldots, u_t)$, by definition 2.5.2 and definition 2.5.5 it follows that

$$\dim \mathcal{T}^{u^j} \ge \dim \zeta(u^j) \qquad \text{for all } j \le t$$

indeed, by unimodularity of $\mathcal{T}$, each simplex $W \in \mathcal{T}$ of codimension 1 is contained in a rational hyperplane.

Definition 2.5.7. A triangulation $\mathcal{T}$ is said **u**-good if

$$\dim \mathcal{T}^{u^j} = \dim \zeta(u^j) \qquad \text{for all } \; j \leq t$$

Given a function $f \in M_n$, a triangulation $\mathcal{T}$ is said f-good if f is linear over each simplex of $\mathcal{T}$. A triangulation $\mathcal{T}$ which is **u**-good and f-good is said $\mathbf{u}f$-good.

Lemma 2.5.8. *Let $\mathbf{u} = (u_0, \ldots, u_t)$ be an index, then:*

(i) For every triangulation $\mathcal{T}$, $\mathcal{T}^{u_j}$ is a face of $\mathcal{T}^{u_{j+1}}$, in symbols

$$\mathcal{T}^{u_j} \preccurlyeq \mathcal{T}^{u_{j+1}}$$

*(ii) Every triangulation $\mathcal{T}$ can be refined to a **u**-good triangulation.*

*(iii) If $\mathcal{W}$ is a refinement of a **u**-good triangulation $\mathcal{T}$, then $\mathcal{W}^{\mathbf{u}} \subseteq \mathcal{T}^{\mathbf{u}}$ and $\mathcal{T}^{\mathbf{u}}$ is the smallest simplex of $\mathcal{T}$ containing $\mathcal{W}^{\mathbf{u}}$.*

*(iv) Every refinement of a **u**-good triangulation (resp., $\mathbf{u}f$-good) is **u**-good (resp., $\mathbf{u}f$-good).*

(v) The following identity holds

$$J_{\mathbf{u}} = \{f \in M_n \mid f|_{\mathcal{T}^{\mathbf{u}}} = 0, \textit{ for some } \mathbf{u}f\textit{-good triangulation } \mathcal{T}\} \tag{2.16}$$

(vi) If $f \in J_{\mathbf{u}}$ then $f|_{\mathcal{U}^{\mathbf{u}}} = 0$, for every $\mathbf{u}f$-good triangulation $\mathcal{U}$.

Proof. *(i)*. It is a direct consequence of definition 2.5.2.
(ii). It follows by lemma 2.1.12.
(iii). Suppose that T is the smallest simplex of $\mathcal{T}$ containing $\mathcal{W}^{\mathbf{u}}$ and suppose that $\mathcal{T}^{\mathbf{u}} \neq T$ (absurdum hypothesis). Since both $\mathcal{T}^{\mathbf{u}}$ and T are simplexes of $\mathcal{T}$ containing some **u**-simplex, by proposition 2.1.15, $\mathcal{T}^{\mathbf{u}} \cap T$ is a simplex of $\mathcal{T}$ containing some **u**-simplex. By minimality of $\mathcal{T}^{\mathbf{u}}$, T strictly contains $\mathcal{T}^{\mathbf{u}}$ and, by minimality of T, $\mathcal{T}^{\mathbf{u}}$ does not contain $\mathcal{W}^{\mathbf{u}}$. Let $S = \mathcal{T}^{\mathbf{u}} \cap \mathcal{W}^{\mathbf{u}}$, again by proposition 2.1.15, S contains some **u**-simplex R. Furthermore, since $\mathcal{W}$ refines $\mathcal{T}$, S is simplex of $\mathcal{W}$ and $\emptyset \neq S \subset \mathcal{W}^{\mathbf{u}}$. This is in contradiction with the minimality of $\mathcal{W}^{\mathbf{u}}$. Then, $T = \mathcal{T}^{\mathbf{u}}$ and $\mathcal{W}^{\mathbf{u}} \subseteq \mathcal{T}^{\mathbf{u}}$.

(iv). Let $\mathcal{U}$ be a refinement of a $\mathbf{u}$-good triangulation $\mathcal{T}$. We know that $\dim \mathcal{U}^j \geq \dim \zeta(u^j)$, for each $j = 0, \ldots, t$. From *(iii)* it follows that $\mathcal{U}^j \subseteq \mathcal{T}^j$, whence $\dim U^j \leq \dim \mathcal{T}^j = \dim \zeta(\mathbf{u})$. Hence $\mathcal{U}$ is $\mathbf{u}$-good.

(v). Consider $f \in \{g \in M_n \mid g|_{\mathcal{T}^{\mathbf{u}}} = 0\}$. Then, the zeroset $f^{-1}(0)$ contains some $\mathbf{u}$-simplex, whence $f \in J_{\mathbf{u}}$. Conversely, suppose that $f \in J_{\mathbf{u}}$. Then there is a unimodular triangulation $\mathcal{T}$ such that f is linear over each simplex of $\mathcal{T}$ (by theorem 2.1.16) and f vanishes over a $\mathbf{u}$-simplex R which is contained in some simplex of $\mathcal{T}$. By *(ii)*, $\mathcal{T}$ can be refined to a $\mathbf{u}$-good triangulation. Let $\mathcal{U}$ be the $\mathbf{u}$-good refinement of $\mathcal{T}$, then R is contained in a union of simplexes of $\mathcal{U}$ and each of these simplexes contains a $\mathbf{u}$-simplex. Then $\mathcal{U}^{\mathbf{u}} \subseteq R$, hence $f|_{\mathcal{U}^{\mathbf{u}}} = 0$.

(vi). By *(v)* there exists at least one $\mathbf{u}f$-good triangulation $\mathcal{T}$ such that $f|_{\mathcal{T}^{\mathbf{u}}} = 0$. Let $\mathcal{U}$ be an arbitrary $\mathbf{u}f$-good triangulation. Then, by lemma 2.1.12 there exists a joint refinement $\mathcal{V}$ of $\mathcal{T}$ and $\mathcal{U}$. Then by *(iii)* and *(iv)*, $V^{\mathbf{u}}$ is subset of $\mathcal{T}^{\mathbf{u}} \cap \mathcal{U}$ having the same dimension of $\mathcal{U}^{\mathbf{u}}$. Thus $f|_{V^{\mathbf{u}}} = 0$. Since f is linear over $\mathcal{U}^{\mathbf{u}}$, $f|_{\mathcal{U}^{\mathbf{u}}} = 0$. □

Definition 2.5.9. Let $\mathbf{u} = (u_0, \ldots, u_t)$ and $\mathbf{v} = (v_0, \ldots, v_r)$ be indexes, with $0 \leq t \leq r \leq n$. If $u_i = v_i$ for each $i = 0, \ldots, t$ then $\mathbf{v}$ is called an *extension* of $\mathbf{u}$. Moreover, if $\zeta(u^t) \subset \zeta(v^r)$, $\mathbf{v}$ is called a *proper extension* of $\mathbf{u}$.

Lemma 2.5.10. *If* $\mathbf{v}$ *is an extension of* $\mathbf{u}$*, then* $J_{\mathbf{v}} \subseteq J_{\mathbf{u}}$.

Proof. Let $f \in J_{\mathbf{v}}$, then by lemma 2.5.8(vi) for any $\mathbf{v}f$-good triangulation $\mathcal{T}$, f vanishes over $\mathcal{T}^{\mathbf{v}}$. It is clear that $\mathcal{T}$ is also $\mathbf{u}f$-good and $\mathcal{T}^{\mathbf{u}} \subseteq \mathcal{T}^{u^t} \subseteq \mathcal{T}^{\mathbf{v}}$. Thus f vanishes over $\mathcal{T}^{\mathbf{u}}$, whence by lemma 2.5.8(*v*) $f \in J_{\mathbf{u}}$. □

We can observe that if we consider an index $\mathbf{u} = (u_0, \ldots, u_t)$ such that $n > t$, it may happen that $t < \dim \zeta(\mathbf{u}) \leq n$. For example, if $\mathbf{u} = u_0$ and $u_0 \notin \mathbb{Q} \cap [0,1]^n$, then $\dim \zeta(u) > 0$. In this case there is an element $v \in \lambda(\mathbf{u})$ such that $(u_1, \ldots, u_t, v)$ is a proper extension.

Definition 2.5.11. Given a triangulation $\mathcal{T}$ and a simplex $F \in \mathcal{T}$, the *star of* F *in* $\mathcal{T}$ is the smallest subcomplex of $\mathcal{T}$ containing all simplexes of $\mathcal{T}$ that contain F and it is denoted by

$$st(F;\mathcal{T})$$

The point-set-theoretical union of $st(F;\mathcal{T})$ is called *closed star* and it is denoted by

$$clstar(F;\mathcal{T})$$

The interior of $clstar(F:\mathcal{T})$ relative to n-cube is called the *open star of F in $\mathcal{T}$* and it is denoted by

$$ostar(F;\mathcal{T})$$

then, it follows that

$$ostar(F;\mathcal{T}) = int\{x \in [0,1]^n \mid \exists n\text{-dimensional } T \in \mathcal{T} \text{ such that } x \in T \supseteq F\}$$

Definition 2.5.12. Given a prime ideal J of M_n, the *germinal ideal* of J, denoted by $germ(J)$, is the intersection of all prime ideals contained in J, i.e.

$$germ(J) = \bigcap\{I \subseteq M_n \mid I \text{ is a prime ideal of } Free_n \text{ and } I \subseteq J\}$$

Theorem 2.5.13. *Given an $\mathbf{u} = (u_0, \ldots, u_t)$ and a function $f \in M_n$, the following conditions are equivalent:*

(i) $f|_{ostar(\mathcal{T}^{\mathbf{u}};\mathcal{T})} = 0$ *for some $\mathbf{u}f$-good triangulation $\mathcal{T}$;*

(ii) $f|_{ostar(\mathcal{T}^{\mathbf{u}};\mathcal{T})} = 0$ *for all $\mathbf{u}f$-good triangulations $\mathcal{T}$;*

(iii) $f \in germ(J_{\mathbf{u}})$.

Proof. *(ii)* $\to$ *(i)*. It is trivial, in fact at least one $\mathbf{u}f$-good triangulation exists.
(iii) $\to$ *(ii)*. Let $\mathcal{T}$ be a $\mathbf{u}f$-good triangulation. By lemma 2.5.8, if $f \in germ(J_{\mathbf{u}}) \subseteq J_u$ then $f|_{\mathcal{T}^{\mathbf{u}}} = 0$. We know that, by definition of $\mathcal{T}^{\mathbf{u}}$, there exist real numbers $\epsilon_1, \ldots, \epsilon_t > 0$ such that

$$conv\{u_0, u_0 + \epsilon_1 u_1, \ldots, u_0 + \epsilon_1 u_1 + \cdots + \epsilon_t u_t\} \subseteq \mathcal{T}^{\mathbf{u}}$$

Suppose $f(x) > 0$ for some $x \in ostar(\mathcal{T}^{\mathbf{u}};\mathcal{T})$ (absurdum hypothesis). Then there is a vector $v \in \lambda(u)^{\perp}$ such that, for all suitably small $\delta > 0$, the function f is linear and non constantly zero over the set

$$R = conv\{u_0, u_0 + \epsilon_1 u_1, \ldots, u_0 + \epsilon_1 u_1 + \cdots + \epsilon_t u_t, u_0 + \epsilon_1 u_1, \ldots, \epsilon_t u_t + \delta v\}$$

Then $f > 0$ over $relint(R)$, if we denote with $(\mathbf{u}, v)$ the $(t+2)$-uple $(u_0, \ldots, u_t, v)$, then $f \notin J_{(\mathbf{u},v)}$ otherwise f vanishes over some $(\mathbf{u}, v)$-simplex Q which can be assumed to be contained in R (by 2.1.15). From lemma 2.5.10 it follows that $J_{(\mathbf{u},v)} \subseteq J_{\mathbf{u}}$ and by proposition 2.5.4 $J_{(\mathbf{u},v)}$ is prime. Whence $J_{(\mathbf{u},v)} \subseteq germ(J_{\mathbf{u}})$, hence $f \notin germ(J_{\mathbf{u}})$, a contradiction.

(i) $\to$ *(iii)*. Suppose $f|_{ostar(\mathcal{T}^{\mathbf{u}};\mathcal{T})} = 0$ for some $\mathbf{u}f$-good triangulation $\mathcal{T}$. Since $\mathcal{T}^{\mathbf{u}} \subseteq ostar(\mathcal{T}^{u}; \mathcal{T})$, $f|_{\mathcal{T}^{\mathbf{u}}} = 0$ and, by 2.5.8, $f \in J_{\mathbf{u}}$. Let J be a prime ideal of M_n such that $J \subseteq J_{\mathbf{u}}$ and suppose $f \notin J$ (absurdum hypothesis). Let $\mathcal{W}$ be a refinement of $\mathcal{T}$ obtained via starring $\mathcal{T}$ at the mediant of $\mathcal{T}^{\mathbf{u}}$: $\mathcal{W}$ has a new vertex b which is obtained by writing each vertex $(v_1/v, \ldots, v_n, v)$ of $\mathcal{T}^{\mathbf{u}}$ in homogeneous coordinates as $(v_1, \ldots, v_n, v)$, then taking the sum $(s_1, \ldots, s_n, s)$ of these vectors, and finally letting $b = (s_1/s, \ldots, s_n/s)$. The vertex $b \in [0,1]^n \cap \mathbb{Q}^n$ is called the Farey mediant of the vertices of $\mathcal{T}^{\mathbf{u}}$. The new refinement $\mathcal{W}$ is automatically unimodular, $\mathbf{u}$-good and $b \in relint(\mathcal{T}^{\mathbf{u}})$. By lemma 2.1.17, we can consider the function $g \in Free_n$ obtained by specifying its valued at vertices of $\mathcal{W}$ as follows:

$$g(x) = \begin{cases} 1 & \text{if } x = b \\ 0 & \text{if } x \text{ is any other vertex of } \mathcal{W} \end{cases}$$

Then, by lemma 2.5.8, $g \notin J_u$, whence $g \notin J$. By construction the function g vanish over the complement of $ostar(\mathcal{T}^{\mathbf{u}}; \mathcal{T})$ in $[0,1]^n$. Then $f \wedge g = 0 \in J$, in contradiction with the primeness of J. $\square$

Proposition 2.5.14. *Given an index* $\mathbf{u} = (u_0, \ldots, u_t)$ *such that* $\dim \zeta(\mathbf{u}) < n$ *and a prime ideal* J *such that* $J \subseteq J_{\mathbf{u}}$*, if there does not exists a proper extension* $\mathbf{v}$ *of* $\mathbf{u}$ *such that* $J \subseteq J_{\mathbf{v}}$ *then there is a function* $f \in J$ *and a* $\mathbf{u}f$*-good triangulation* $\mathcal{T}$ *such that*

(i) $f|_{\mathcal{T}^{\mathbf{u}}} = 0$

(ii) $f(x) > 0$ *for all* $x \in clstar(\mathcal{T}^{\mathbf{u}}; \mathcal{T}) \setminus \mathcal{T}^{\mathbf{u}}$

Proof. Let $\zeta^{\perp}$ denote the affine space given by u_0-translation of $\lambda(\mathbf{u})^{\perp}$. Suppose d is the dimension of $\zeta^{\perp}$ then $q = n - \dim \zeta(\mathbf{u})$. Let S be the $(q-1)$-dimensional sphere of radius one, centered in u_0 and lying in $\zeta^{\perp}$, in symbols

$$S = \{z \in \zeta^{\perp} \mid d(z, u_0) = 1\}$$

Given an arbitrary unit vector $v \in \lambda(\mathbf{u})^{\perp}$, the index $(\mathbf{u}, v)$ is a proper extension of $\mathbf{u}$. Since there does not exist a proper extension $\mathbf{v}$ of $\mathbf{u}$ such that $J \subseteq J_{\mathbf{v}}$, $J \nsubseteq J_{(\mathbf{u},v)}$. Then there exists $f_v \in J \setminus J_{(\mathbf{u},v)}$. Since $J \subseteq J_{\mathbf{u}}$, $f_v \in J_{\mathbf{u}}$. Let $\mathcal{T}_v$ be a $(\mathbf{u}v)f$-good triangulation. Trivially $\mathcal{T}_v$ is $\mathbf{u}f$-good, then

$$f_v|_{\mathcal{T}_v^{\mathbf{u}}} = 0 \qquad \text{and} \qquad f_v(x) > 0 \text{ for all } x \in relint(\mathcal{T}_v^{(\mathbf{u},v)}) \tag{2.17}$$

Denote with O_v the open star of $\mathcal{T}_v^{(\mathbf{u},v)}$ in $\mathcal{T}_v$, it follows that

$$f_v(x) > 0 \qquad \forall x \in O_v \tag{2.18}$$

One can observe that f_v is linear over each n-simplex of the star of $\mathcal{T}_v^{(\mathbf{u},v)}$ in $\mathcal{T}_v$ and is > 0 over $relint(\mathcal{T}_v^{(\mathbf{u},v)}) \subseteq O_v$. We denote with O'_v the projection of O_v into $\zeta^{\perp}$. The set O'_v is relatively open in $\zeta^{\perp}$ because is a projection of a open set. For each $y \in O'_v$ we denote with $\hat{y}$ the intersection of the sphere S with the half-line originating in u_0 and passing through y. Then the set

$$\hat{O}_v = \{\hat{y} \mid y \in O'_v\}$$

is relatively open in the sphere S. If the unit vector v ranges over all unit vectors of $\lambda(\mathbf{u})^{\perp}$, we obtain a family

$$\mathcal{O} = \{\hat{O}_v \mid v \in \lambda^{\perp}\}$$

which is a open cover of S. Since S is compact, there is a finite subfamily

$$\{\hat{O}_{v(1)}, \hat{O}_{v(2)}, \ldots, \hat{O}_{v(k)}\}$$

of $\mathcal{O}$ which is still a cover of S. For each $v(i)$ we have a function $f_i = f_{v(i)} \in J \setminus J_{(\mathbf{u},v(i))}$ and some $(\mathbf{u}, v(i))$-good triangulation $\mathcal{T}_i = \mathcal{T}_{v(i)}$ such that the conditions expressed in (2.17) are satisfied.

Claim 1. For each non zero vector $w \in \lambda^{\perp}$ there is $i \in \{1, \ldots, k\}$ such that the closed star of $\mathcal{T}_i^{(\mathbf{u},v(i))}$ in $\mathcal{T}_i$ contains some $(\mathbf{u}, w)$-simplex

$$conv\{u_0, u_0 + \epsilon_1 u_1, \ldots, u_0 + \epsilon_1 u_1 + \cdots + \epsilon w\}$$

whose vertex $u_0 + \epsilon_1 u_1 + \cdots + \epsilon w$ lies in $O_{v(i)}$.

Proof claim 1. Let $x = u_0 + w \in \zeta^{\perp}$. As we have seen previously, $\hat{x}$ denotes the intersection of S with the half-line originating in u_0 passing through x. Since the family

$\{\hat{O}_{v(1)}, \hat{O}_{v(2)}, \ldots, \hat{O}_{v(k)}\}$ is an open cover of S, there exists a $v(i)$ and $y \in O'_{v(i)}$ such that y coincides with $u_0+\delta w$, for some $\delta > 0$. Then, there is a point $z = u_0+\epsilon_1 u_1+\cdots+\epsilon w \in O_{v(i)}$ whose projection into $\zeta^{\perp}$ coincides with y. By definition of $O_{v(i)}$ there is a n-simplex R in the star of $\mathcal{T}_i^{(\mathbf{u},v(i))}$ such that $z \in R$. Since R is convex and $\mathcal{T}_i^{\mathbf{u}}$ is a proper subface of R, it follows that

$$conv\{u_0, u_0+\epsilon_1 u_1, \ldots, u_0+\epsilon_1 u_1+\cdots+\epsilon_t u_t+\epsilon w\} \subseteq R \subseteq clstar(\mathcal{T}_i^{(\mathbf{u},v(i))}; \mathcal{T}_i)$$

Then the claim is settled.

Consider the function $f \in J$ defined by

$$f = f_1 \vee f_2 \vee \cdots \vee f_k \tag{2.19}$$

where f_i is the function associated to the vector $v(i)$, for each $i = 1, \ldots, k$. In the light of lemma 2.1.12, there exists a jointly f-good refinement $\mathcal{T}$ of the family $\mathcal{T}_1, \ldots, \mathcal{T}_k$. By (2.17) and lemma 2.5.8, $f_i|_{\mathcal{T}^{\mathbf{u}}} = 0$, for each $i = 1, \ldots, k$. Then, it follows that

$$f|_{\mathcal{T}^{\mathbf{u}}} = 0$$

Claim 2. $f(x) > 0$ for each $x \in clstar(\mathcal{T}^{\mathbf{u}}; \mathcal{T}) \setminus \mathcal{T}^{\mathbf{u}}$.

Proof claim 2. First of all, assume $x \in ostar(\mathcal{T}^{\mathbf{u}}; \mathcal{T}) \setminus \mathcal{T}^{\mathbf{u}}$. Then $x \in relint(T)$ for a uniquely determined smallest simplex $T \in \mathcal{T}$ in the star of $\mathcal{T}^{\mathbf{u}}$. It follows that $\mathcal{T}^{\mathbf{u}}$ is a proper face of T, whence $\dim T > \dim \mathcal{T}^{\mathbf{u}}$. The vector $x - u_0$ con be uniquely written as $x - u_0 = l + v$ where $l \in \lambda(\mathbf{u})$ and $v \in \lambda(\mathbf{u})^{\perp}$. Since $x \notin \mathcal{T}^{\mathbf{u}}$, we have $v \neq 0$. Then, the simplex T contains some $(\mathbf{u}, v)$-simplex. Denoting with $\bar{O}$ the closure of the set O, by **Claim 1**, the closed star $\bar{O}_{v(i)}$ of $\mathcal{T}_i^{(\mathbf{u},v(i))}$ in $\mathcal{T}_i$ contains some $(\mathbf{u}, v)$-simplex. Then, by proposition 2.1.15, the simplex $T \cap \bar{O}_{v(i)}$ contains some $(\mathbf{u}, v)$-simplex

$$T' = conv\{u_0, u_0+\epsilon_1 u_1, \ldots, u_0+\omega_1 u_1+\cdots+\omega_t u_t+\omega v\} \subseteq T \cap \bar{O}_{v(i)}$$

Let $c \in relint(T')$. Then $c \in O_{v(i)}$ and from (2.18) it follows that $f_i(c) > 0$. Since $f \geq f_i > 0$ over $O_{v(i)}$ it follows that $f(c) > 0$. Since $c \in relint(T)$ and $\mathcal{T}$ is f-good, $f > 0$ over $relint(T)$. Thus $f(x) > 0$ for all $x \in relint(T)$. Our claim is settled in the case when $x \in ostar(\mathcal{T}^{\mathbf{u}}; \mathcal{T}) \setminus \mathcal{T}^{\mathbf{u}}$.

Assume $x \in clstar(\mathcal{T}^{\mathbf{u}};\mathcal{T}) \setminus \mathcal{T}^{\mathbf{u}}$. Then we can find a point $y \in relint(\mathcal{T}^{\mathbf{u}})$ (for example y can be chosen as the Farey mediant of the vertices of $\mathcal{T}^{\mathbf{u}}$) such that the segment joining x and y contains some point $z \in ostar(\mathcal{T}^{\mathbf{u}};\mathcal{T}) \setminus \mathcal{T}^{\mathbf{u}}$. This segment is contained in some simplex of the star of $\mathcal{T}^{\mathbf{u}}$ and f is linear over all simplexes in $\mathcal{T}$. Since $y \in relint(\mathcal{T}^{\mathbf{u}})$, $f(y) = 0$ and, by previous discussion, $f(z) > 0$. Then $f(x) > 0$ and the claim is settled.

□

Theorem 2.5.15. *Given and index* $\mathbf{u} = (u_0, \ldots, u_t)$ *and a prime ideal* J *such that* $J \subseteq J_{\mathbf{u}}$*, if there exists no proper extension* $\mathbf{v}$ *of* $\mathbf{u}$ *such that* $J \subseteq J_{\mathbf{v}}$*, then* $J = J_{\mathbf{u}}$*.*

Proof. **Case 1**. If $\dim \zeta(\mathbf{u}) = n$, then there is no vector $v \in \mathbb{R}^n$ such that the index $(\mathbf{u}, v)$ is a proper extension of $\mathbf{u}$. Suppose that $J \subset J_{\mathbf{u}}$ (absurdum hypothesis), then there exists $f \in J_{\mathbf{u}} \setminus J$ and by theorem 2.1.16 there exists a f-good triangulation $\mathcal{V}$ which can be refined to a $\mathbf{u}f$-good triangulation $\mathcal{T}$ by lemma 2.5.8(ii). Since the zeroset of f contains some $\mathbf{u}$-simplex, $\mathcal{T}^{\mathbf{u}}$ is given by the intersection of all simplexes of $\mathcal{T}$ which contain some $\mathbf{u}$-simplex an $\mathcal{T}$ is $\mathbf{u}f$-good, $\dim \mathcal{T}^{\mathbf{u}} = n$ and $f|_{\mathcal{T}^{\mathbf{u}}} = 0$. As we have done in theorem 2.5.13, let $\mathcal{W}$ be the refinement of $\mathcal{T}$ obtained by starring $\mathcal{T}$ at the mediant b of $\mathcal{T}^{\mathbf{u}}$. In the light of lemma 2.1.17 let $g \in Free_n$ be the function determined by specifying its value at each vertex of $\mathcal{W}$ as follows

$$g(x) = \begin{cases} 1 & \text{if } x = b \\ 0 & \text{if } x \text{ is any other vertex of } \mathcal{W} \end{cases}$$

with g linear over each simplex of $\mathcal{W}$. Since $f|_{\mathcal{T}^{\mathbf{u}}} = 0$, we have $g \wedge f = 0$, whence $g \wedge f \in J$. By construction $g \notin J_{\mathbf{u}}$. Since $f \notin J$ it follows that J is not prime, a contradiction.

Case 2. If $\dim \zeta(\mathbf{u}) < n$, consider g arbitrary function in $J_{\mathbf{u}}$. By proposition 2.5.14 there exists a function $f \in J$ and a $\mathbf{u}f$-good triangulation $\mathcal{T}$ satisfying the conditions therein. Our aim is to construct a function $h \in J$ such that g is in the ideal generated by $f \oplus h$ which is contained in the ideal J, thus showing that $J_{\mathbf{u}} = J$. An application of lemma 2.1.12 yields a $\mathbf{u}fg$-good triangulation $\mathcal{V}$ which refines $\mathcal{T}$. Since $g \in J_{\mathbf{u}}$, then $g|_{\mathcal{T}}^{\mathbf{u}} = 0$ because $\mathcal{T}^{\mathbf{u}}$ is given by the intersection of all simplex in $\mathcal{T}$ which contain some $\mathbf{u}$-simplex and g vanishes over some $\mathbf{u}$-simplex. By lemma 2.5.8, $\mathcal{V}^{\mathbf{u}} \subseteq \mathcal{T}^{\mathbf{u}}$ therefore $g|_{\mathcal{V}^{\mathbf{u}}} = 0$. In the light of *lemma* 2.1.17 let $h \in Free_n$ the function

$$h(x) = \begin{cases} 0 & \text{if } x \text{ is a vertex of some simplex in the star of } \mathcal{V}^{\mathbf{u}} \\ 1 & \text{if } x \text{ is any other vertex of } \mathcal{V} \end{cases}$$

with h linear over each simplex of $\mathcal{V}$. Then h vanishes over $ostar(\mathcal{V}^{\mathbf{u}}; \mathcal{V})$, whence by theorem 2.5.13 $h \in germ(J_{\mathbf{u}})$. Since by our hypothesis J is prime and $J \subseteq J_{\mathbf{u}}$ it follows that $h \in J$. We can observe that proposition 2.5.14 continues to be satisfied by every refinement of the triangulation $\mathcal{T}$. In fact by lemma 2.5.8, $\mathcal{V}^{\mathbf{u}} \subseteq \mathcal{T}^{\mathbf{u}}$, then f vanishes over $\mathcal{V}^{\mathbf{u}}$. Thus the condition (i) is easily satisfied by $\mathcal{V}$. First of all, in order to prove that $\mathcal{V}$ satisfied also the second condition, we can observe that if $\mathcal{V}^{\mathbf{u}} = \mathcal{T}^{\mathbf{u}}$ the conclusion follows trivially. If $\mathcal{V}^{\mathbf{u}} \subset \mathcal{T}^{\mathbf{u}}$ it follows that $\mathcal{T}^{\mathbf{u}}$ can not be a simplex of $\mathcal{V}$ because $\mathcal{V}^{\mathbf{u}}$ and $\mathcal{T}^{\mathbf{u}}$ have the same dimension (since both of them are **u**-good) and $\mathcal{V}$ is unimodular. Therefore, $\mathcal{T}^{\mathbf{u}}$ is divided by the operation of refinement which yields $\mathcal{V}$ and each part is a simplex of the refinement $\mathcal{V}$. One of these part is the simplex $\mathcal{V}^{\mathbf{u}}$ and the other parts of $\mathcal{T}^{\mathbf{u}}$ are not in the closed star of $\mathcal{V}^{\mathbf{u}}$ in $\mathcal{V}$. Whence the only points of the $clstar(\mathcal{V}^{\mathbf{u}}; \mathcal{V})$ where f vanish are those of $\mathcal{V}^{\mathbf{u}}$. Hence the condition (ii) of proposition 2.5.14 is satisfied by $\mathcal{V}$. Therefore, $(f \oplus h)(x) = 0$ if and only if $x \in \mathcal{V}^{\mathbf{u}}$. Since $g|_{\mathcal{V}^{\mathbf{u}}} = 0$, the zeroset of $f \oplus g$ is contained in the zeroset of g, therefore by lemma 2.2.9 $g \in \langle f \oplus h \rangle \subseteq J$. Hence $J_{\mathbf{u}} = J$. □

Corollary 2.5.16. *Every prime ideal J of M_n has the form $J = J_{\mathbf{u}}$ for some index* $\mathbf{u}$.

Proof. Every prime ideal of M_n is contained in exactly one maximal ideal. As we have seen in theorem 2.2.4, all maximal ideal of M_n are exactly those of the form $J_x = \{f \in M_n \mid f(x) = 0\}$, for some $x \in [0,1]^n$. In other words, if we consider an index $\mathbf{u} = (u_0, u_1, \dots, u_t)$, J_{u_0} is a maximal ideal, (u_0, u_1) is an index and it is an extension of the index u_0, therefore by lemma 2.5.10 $J_{(u_0,u_1)} \subseteq J_{u_0}$. Therefore, iterating this process, we obtain a chain of prime ideals

$$J_{u_0} \supseteq J_{(u_0,u_1)} \supseteq J_{(u_0,u_1,u_2)} \supseteq \cdots \supseteq J_{(u_0,u_1,\dots,u_t)} \tag{2.20}$$

Suppose that $\mathbf{u}$ is an index such that $J_{\mathbf{u}} \supseteq J$ and there does not exits a proper extension $\mathbf{v}$ of $\mathbf{u}$ such that $J_{\mathbf{v}} \supseteq J$. Then, by theorem 2.5.15, $J = J_{\mathbf{u}}$. □

Chapter 3

Strong completeness in Łukasiewicz propositional logic Ł$_\infty$

In this chapter we shall tackle the problem of *completeness* in Łukasiewicz propositional logic. As we shall see for the tautologies, the completeness theorem is satisfied, in other words the set of semantic tautologies coincides with the set of syntactic tautologies. The situation is very different if we consider the deductive closure of a set of formulas Θ. In fact, in general, the set of semantic consequences does not coincide with the set of syntactic consequences. We shall show an example of a formula which is a semantic consequence of a family of formulas Θ but it is not a syntactic consequence. This example highlights the inadequacy of the Bolzano-Tarski paradigm, i.e. the usual definition of valuation. Subsequently we shall establish those cases in which the completeness is satisfied. In order to do this, following [9], we shall associate the two sets of semantic and syntactic consequences with two filters in the free MV-algebra and we shall see what conditions must be satisfied for these two filters to coincide. In the last part of the chapter we shall define a new concept of valuation giving a new notion of satisfiability in which the prime ideals will come into play. As done in [10], we shall call these new valuations *differential valuations* as they take into account the differential properties of McNaughton functions.

3.1 Semantic consequence relation in Ł$_\infty$

By McNaughton's theorem 2.1.6 and proposition 1.6.1, for every formula p, the equivalence class $[p]$ can be identified with a function $f_p\colon [0,1]^\omega \to [0,1]$ in the MV-algebra of McNaughton's functions. We denote with VAL the set of valuations of formulas in $FORM$. It can be identified with the Hilbert cube $[0,1]^\omega$ via the restriction map

$$v \in VAL\colon \mapsto V = v_{|\{X_0,X_1,\dots\}} \in [0,1]^{\{X_0,X_1,\dots\}} = [0,1]^\omega$$

Then for each $p \in FORM$ we have $v(p) = f_p(V)$, where V is now realized as the element $(V_0, V_1, \dots) \in [0,1]^\omega$ such that $V_n = V(X_n)$. From these observations we can reformulate the definition of *semantic consequence* as follows.

Definition 3.1.1. In Ł$_\infty$, given a set $\Theta \subseteq FORM$ it is possible to define the relation $\models$ of semantic consequence, for all $p \in FORM$, by stipulating that:

$$\Theta \models p \text{ iff } \forall V \in [0,1]^\omega,\ (f_q(V) = 1 \text{ for all } q \in \Theta \implies f_p(V) = 1)$$

On the other hand, the set of syntactic consequences of Θ, denoted with $\Theta^\vdash$, is the smallest subset of $FORM$ closed under modus ponens, containing Θ and tautologies.

Remark 3.1.2. What has been observed previously can be reported to the case in which the set of formulas is constructed from a finite set of $n \geq 1$ variables. In this case we denote with VAL_n the set of all valuations of formulas in $FORM_n$ and it can be identified with the unit cube $[0,1]^n$ as we have seen previously.

Lemma 3.1.3. *Each formula provable from a set $\Theta \subseteq FORM$ of formulas is also a semantic consequence of this set. In other words, the following inclusion holds*

$$\Theta^\vdash \subseteq \Theta^\models \tag{3.1}$$

In particular, all provable formulas are tautologies.

Proof. Let $v \in VAL$ be a valuation such that $v(\alpha) = 1$ for all $\alpha \in \Theta$. By induction on n we shall prove that if $\alpha_1, \alpha_2, \dots, \alpha_n$ is a proof from Θ then $v(\alpha_n) = 1$. If $n = 1$ then α_1 is an axiom or it belongs to Θ. In the first case, observing that all axioms are tautologies, we have

$v(\alpha_1) = 1$. In the second case by the hypothesis on v we have $v(\alpha_1) = 1$. Assume $n > 1$ and suppose that, for each proof from Θ, $\beta_1, \ldots, \beta_m$, with $m < n$, $v(\beta_m) = 1$. Let $\alpha_1, \ldots, \alpha_n$ be a proof from Θ. If α_n is not an axioms and $\alpha_n \notin \Theta$ then there are $i, j \in \{1, \ldots, n\}$ such that α_j is the formula $(\alpha_i \to \alpha_n)$. Since both $\alpha_1, \ldots, \alpha_i$ and $\alpha_1, \ldots, \alpha_j$ are proofs from Θ, by induction hypothesis we have $v(\alpha_i) = v(\alpha_j) = 1$. Therefore

$$1 = v(\alpha_j) = 1 \to v(\alpha_n) = v(\alpha_n)$$

□

The converse inclusion is verified for the set of *tautologies* and it follows from Chang's completeness theorem. Recalling the definition of the *syntactic equivalence* $\equiv$ and Lindenbaum algebra L, we can prove the following result which states that the set of semantic tautologies $\emptyset^{\models}$ coincides with the set of syntactic tautologies $\emptyset^{\vdash}$.

Theorem 3.1.4. *Every tautology is provable, in symbols*

$$\emptyset^{\models} = \emptyset^{\vdash} \tag{3.2}$$

Proof. For each propositional variable X_i, the class $[X_i]$ is an element of the Lindebaum algebra L. Let $\alpha \in FORM$ with $Var(\alpha) \subseteq \{X_{i_1}, \ldots, X_{i_n}\}$. Then by induction on the number of connectives in α we have

$$\alpha^L([X_{i_1}], \ldots, [X_{i_n}]) = [\alpha] \tag{3.3}$$

Thus, if $\alpha \in FORM$ is not provable, then $[\alpha] \neq 1$, whence $\alpha^L([X_{i_1}], \ldots, [X_{i_n}]) \neq 1$. In other words, the Lindenbaum algebra L does not satisfy the equation $\alpha = 1$. Then, by Chang's completeness theorem 1.3.5, the MV-algebra [0,1] does not satisfy the equation $\alpha = 1$, i.e. α is not a tautology. □

Differently by the classical logic, the traditional semantic relation $\models$ in Ł$_\infty$ fails to be strongly complete. In lemma 3.1.3 we have proved the inclusion $\Theta^{\vdash} \subseteq \Theta^{\models}$ but, as we shall see, in general, $\Theta^{\models} \neq \Theta^{\vdash}$. The differential properties of f_p, for all $p \in \Theta$ are ignored by the semantic consequence relation $\models$ that we have already defined, although they have no less semantical content than the truth-value $f_p(V)$. The following example involves formulas in one variable and it shows as the Bolzano-Tarski paradigm does not work in Ł$_\infty$.

Example 3.1.5. Suppose $\Theta \subseteq FORM_1$ and suppose Θ is satisfied by a unique valuation $V \in [0,1]$ with $V < 1$ and $V \in \mathbb{Q}$. Suppose that $\partial f_p/\partial X^+(V)=0$ for all $p \in \Theta$. Let $q = q(X)$ be a formula with $f_q(V) = 1$ and $f_q(W) < 1$ for all $W > V$. Intuitively the hypothesis means that each $p \in \Theta$ is also true for all $W > V$ sufficiently close to V. In other words, p is 'stably' true at V. Instead q misses this stability property, although q is a semantic consequence of Θ. It should be noted that $\Theta \nvdash q$, in fact if we suppose $\Theta \vdash p$ (absurdum hypothesis) then, by compactness, there is a finite subset $\{\theta_1, \theta_2, \dots, \theta_k\} \subseteq \theta$ such that $\{\theta_1, \theta_2, \dots, \theta_k\} \vdash p$. By our hypothesis on Θ, each θ_i is stably true at V, but p is only true at V, then $\{\theta_1, \dots, \theta_k\} \not\models p$, in contradiction with lemma 3.1.3. Thus $\Theta \nvdash p$.

Now our aim is to give a necessary and sufficient conditions for $\Theta^\vdash$ coincides with $\Theta^\models$.

Definition 3.1.6. Each set $\Theta \subseteq FORM$ determines a filter F_Θ and an ideal I_Θ in the MV-algebra of McNaughton's function M, defined as follows

1. $F_\Theta = \langle \{f_p \mid p \in \Theta\} \rangle$
2. I_Θ is the ideal of M generated by the set $\{1 - f_p \mid p \in \Theta\}$

which correspond, via the isomorphism between M and L, to the filter $F(\Theta)$ generated by $[\Theta]$ and the ideal $I(\Theta) = F(\Theta)^*$ defined in definition 1.6.7.

Proposition 3.1.7. *(i) Let $\emptyset \neq \Theta \subset FORM$. Then $\Theta^\vdash = \hat{\Theta}$*

(ii) For each $p \in FORM$ we have $p \in \Theta^\vdash$ iff $f_p \in F_\Theta$ iff $f_p \in F_{\hat{\Theta}}$ iff ${f_p}^* \in I_\Theta$

Proof. (i). It is easy to check that $\hat{\Theta}$ is a theory containing Θ. Conversely, let consider $p \in \hat{\Theta}$, then $q_1 \to (q_2 \to \cdots (q_n \to p)$ is a tautology with $q_1, \dots, q_n \in \Theta$. Then by an induction argument it follows that $p \in \Theta^\vdash$. By proposition 1.6.3 (i) $\hat{\Theta} = \Theta^\vdash$. (ii). It follows from (i) and proposition 1.6.8. □

Therefore, by proposition 3.1.7 F_Θ is the set of McNaughton's functions corresponding to the syntactic consequences of Θ. If $\Theta = \emptyset$, then $\Theta^\vdash$ is the set of tautologies and the proposition 3.1.7 can be extended to this case by defining $\hat{\Theta}$ as the set of all tautologies, $F_\Theta = \{1\}$, as done in proposition 1.6.8.

Now consider the set of semantic consequences $\Theta^{\models}$, we shall associate to this set another filter in the MV-algebra of McNaughton's functions M. Previously it is necessary to give the following definitions.

Definition 3.1.8. Given a set $\Theta \subseteq FORM$ with $\Theta \neq \emptyset$, we can define the close set of $[0,1]^\omega$

$$X_\Theta = V_{I_\Theta} = \{x \in [0,1]^\omega \mid f(x) = 0 \text{ for all } f \in I_\Theta\}$$

Then, by the definition of I_Θ and F_Θ and by proposition 3.1.7

$$X_\Theta = \{x \in [0,1]^\omega \mid f_q(x) = 1 \text{ for all } q \in \Theta\} \tag{3.4}$$

We can define the ideal I^Θ and the filter F^Θ as follows:

$$I^\Theta = \{f \in L \mid f = \mathbf{0} \text{ on } X_\Theta\} = J_{X_\Theta} \tag{3.5}$$

$$F^\Theta = \{f \in L \mid f = \mathbf{1} \text{ on } X_\Theta\} = (I_\Theta)^* \tag{3.6}$$

In case $\Theta = \emptyset$, $X_\Theta = [0,1]^\omega$, $I^\Theta = \{0\}$ and $F^\Theta = \{1\}$.

Proposition 3.1.9. *Let $\Theta \subseteq FORM$. Then for all $p \in FORM$*

$$p \in \Theta^{\models} \textit{ iff } f_p \in F^\Theta \tag{3.7}$$

Proof.

$$\begin{aligned} p \in \Theta^{\models} &\text{ iff for all } V \in [0,1]^\omega, \text{ if } f_q(V) = 1 \text{ for every } q \in \Theta \text{ then } f_p(V) = 1 \\ &\text{ iff } f_p = 1 \text{ on } X_\Theta \\ &\text{ iff } f_p \in F^\Theta \end{aligned}$$

□

Therefore the filter F^Θ is the filter given by all McNaughton functions corresponding to semantic consequences of Θ.

The following theorem gives a necessary and sufficient condition for $\Theta^{\vdash}$ coincides with $\Theta^{\models}$.

Theorem 3.1.10. *For every $\Theta \subseteq FORM$ the following are equivalent:*

(i) The two sets of semantic consequences $\Theta^{\models}$ and syntactic consequences $\Theta^{\vdash}$ coincide.

(ii) The ideal I_Θ satisfies the equivalent conditions of theorem 2.4.1 or $I_\Theta = M$.

Proof.

$$\begin{aligned}
\Theta^{\vdash} = \Theta^{\models} &\text{ iff } I_\Theta = I^{\Theta} \quad \text{by proposition 3.1.7 and proposition 3.1.9} \\
&\text{ iff } I_\Theta = J_{X_\Theta} \\
&\text{ iff } I_\Theta = J_{V_{I_\Theta}}
\end{aligned}$$

Now apply theorem 2.4.1(i) □

Therefore, recalling the definition of *Lindenbaum algebra of* Θ, the following corollary is a direct consequence of theorem 3.1.10.

Corollary 3.1.11. *The set of semantic consequences $\Theta^{\models}$ coincides with the set of syntactic consequences $\Theta^{\vdash}$ if and only if $L(\Theta)$ is semisimple.*

Proof. It follows from theorem 3.1.10 together with theorem 2.4.1(v). □

Remark 3.1.12. From theorem 3.1.10 and theorem 2.4.2 it follows that the identity

$$\Theta^{\vdash} = \Theta^{\models}$$

holds in the following cases:

(i) when Θ is maximally consistent (i.e. I_Θ is maximal);

(ii) when Θ is finite;

(iii) when $\Theta = \Theta_n$ is the infinite set of axioms for MV_n-algebras, for $n > 1$;

(iv) when for each point x in the boundary of X_Θ the set $\{x_0, x_1, \dots\}$ is linearly independent if the **Q**-vectorspace **R**.

3.2 Stable consequence relation

As we have observed the classical notion of semantic consequence of a set Θ of formulas fails to be strongly complete. In this section we shall give a new notion of semantic consequence, that turns out to coincide with syntactic consequence, introducing a new and enriched definition of valuation.

Definition 3.2.1. For $n = 1, 2, \ldots$ and let $\mathbf{u} = (u_0, u_1, \ldots, u_t)$ be a $(t+1)$-uple of elements of $\mathbb{R}^n$ where $u_1, \ldots, u_t$ are pairwise orthogonal unit vectors. For each $m = 1, 2, \ldots$ let the t-simplex $T_{\mathbf{u},\,m} \subseteq \mathbb{R}^n$ be defined as follows

$$T_{\mathbf{u},m} = conv\{u_0,\, u_0 + u_1/m,\, u_0 + u_1/m + u_2/m^2, \ldots, u_0 + u_1/m + u_2/m^2 + \cdots + u_t/m^t\}$$

We say that $\mathbf{u}$ is a *differential valuation* of order t in $\mathbb{R}^n$ if for all large m the t-simplex $T_{\mathbf{u},m}$ is contained in the n-cube $[0,1]^n$.

Proposition 3.2.2. *Let $\mathbf{u} = (u_0, u_1, \ldots, u_t)$ be a differential valuation, then we have*

(i) For all $m = 1,2,\ldots$, $T_{\mathbf{u},m} \supseteq T_{\mathbf{u},m+1}$

(ii) For each $\mathbf{u}$-simplex

$$T = conv\{u_0, u_0 + \epsilon_1 u_1, \ldots, u_0 + \epsilon_1 u_1 + \cdots \epsilon_t u_t\}$$

there is $m = 1, 2, \ldots$ such that $T_{\mathbf{u},m} \subseteq T$.

Proof. (i)-(ii) are easily verified by induction. □

Definition 3.2.3. Let $\mathbf{u} = (u_0, u_1, \ldots, u_t)$ be a differential evaluation, we define the subset $P_{\mathbf{u}}$ of M_n as follows

$$P_{\mathbf{u}} = \{f \in M_n \mid f^{-1}(0) \supset T_{\mathbf{u},m} \text{ for some } m\}$$

Proposition 3.2.4. *Let $\mathbf{u} = (u_0, u_1, \ldots, u_t)$ be a differential valuation.*

(i) The set $P_{\mathbf{u}}$ is a prime ideal of M_n;

(ii) Every prime ideal J of M_n has the form $J = P_{\mathbf{v}}$ for some differential valuation $\mathbf{v}$.

Proof. These two properties follows from proposition 3.2.2 together with proposition 2.5.1 and corollary 2.5.16. □

Remark 3.2.5. Let $\mathbf{u} = (u_0, u_1, \ldots, u_t)$ be a *differential valuation* in $\mathbb{R}^n$, then the prime ideals of M_n can be visualized as follows:

(0) P_{u_0} is the maximal ideal of M_n given by all functions of M_n that vanish at u_0 (theorem 2.2.4);

(1) $P_{(u_0,u_1)}$ is the prime ideal of M_n given by all functions of M_n that vanish on an interval of the form
$$conv\{u_0, u_0 + u_1/m\}$$
for some integer $m > 0$. Equivalently, $P_{\mathbf{u}}$ is given by all functions $f \in M_n$ such that $f(u_0) = 0$ and $\partial f(u_0)/\partial u_1 = 0$;

(2) $P_{(u_0,u_1,u_2)}$ is the prime ideal of M_n given by all functions of M_n that vanish on a set of the form
$$conv\{u_0,\, u_0 + u_1/m,\, u_0 + u_1/m + u_2/m^2\}$$
for some integer $m > 0$. Equivalently $P_{(u_0,u_1,u_2)}$ is given by all functions $f \in M_n$ such that f vanishes on an interval of the form $conv\{u_0, u_0 + u_1/m\}$, for some integer $m > 0$, and $\partial f(y)/\partial u_2 = 0$ for all $y \in relint(conv\{u_0, u_0 + u_1/m\})$.

(t) $P_{(u_0,\ldots,u_t)}$ is the prime ideal of M_n given by all $f \in M_n$ such that, for some integer $m > 0$, f vanishes on the $(t-1)$-simplex
$$T = conv\{u_0, u_0 + u_1/m, \ldots, u_0 + u_1/m + \cdots u_t/m^t\}$$
and $\partial f(y)/\partial u_t = 0$ for all $y \in relint(T)$.

Observe that $P_{u_0} \supseteq P_{(u_0,u_1)} \supseteq \cdots \supseteq P_{(u_0,\ldots,u_t)}$.

Definition 3.2.6. Let $\mathbf{u} = (u_0, u_1, \ldots, u_t)$ be a differential valuation in $\mathbb{R}^n$ and let $\phi(X_1, \ldots, X_n)$ be a formula. We say that $\mathbf{u}$ satisfies ϕ if $1 - f_\phi \in P_{\mathbf{u}}$. Thus
$$f_\phi(u_0) = 1 \qquad \partial f_\phi(u_0)/\partial u_1 = 0$$

and $1 - f_\phi$ satisfies the conditions (2) through (t) in remark 3.2.5.

Definition 3.2.7. Given $\Theta \subseteq FORM_n$ and $\phi \in FORM_n$ we say that ϕ is a *stable consequence* of Θ, in symbols

$$\Theta \models_\partial \phi$$

if ϕ is satisfied by every differential valuation $\mathbf{u} = (u_0, \ldots, u_t)$ that satisfies every $\theta \in \Theta$.

Remark 3.2.8. Observe that the traditional valuation coincides with differential valuation of order 0, therefore $\Theta \models \phi$ if and only if ϕ is satisfied by every differential valuation of order 0 which satisfies every $\theta \in \Theta$. Therefore if $\Theta \models_\partial \phi$ then $\Theta \models \phi$.

Theorem 3.2.9. *Given $\Theta \subseteq FORM_n$ and $\phi \in FORM_n$, then*

$$\Theta \models_\partial \phi \;\Leftrightarrow\; \Theta \vdash \phi$$

Proof. Let $I_\Theta = \langle 1 - f_\theta \mid \theta \in \Theta \rangle$ be the ideal of M_n generated by the functions given by all negations of formulas in Θ.

$\Theta \vdash \phi$ iff $1 - f_\phi \in I_\Theta$ by proposition 3.1.7

iff $1 - f_\phi$ belongs to every prime ideal $P \supseteq I_\Theta$ by corollary 1.1.30

iff $1 - f_\phi$ belongs to every prime ideal P such that $1 - f_\theta \in P \; \forall \theta \in \Theta$

iff for every differential valutation $\mathbf{u}$ in $\mathbb{R}^n$, if $1 - f_\theta \in P_\mathbf{u} \; \forall \theta \in \Theta$ then $1 - f_\phi \in P_\mathbf{u}$, by proposition 3.2.4

iff ϕ is satisfied by all differential valuation $\mathbf{u}$ satisfying all $\theta \in \Theta$, by definition 3.2.6

iff $\Theta \models_\partial \phi$

□

By theorem 3.2.9, we have the following result which states the compactness of $\models_\partial$.

Corollary 3.2.10. *Let $\Theta \subseteq FORM_n$ and $\phi \in FORM_n$. Then*

$$\Theta \models_\partial \phi \;\; \textit{if and only if} \; \{\theta_1, \ldots, \theta_k\} \models_\partial \phi$$

for some finite set $\{\theta_1, \ldots, \theta_k\} \subseteq \Theta$.

Since $FORM_n \subseteq FORM_{n+1}$, it seems that the definition of $\Theta \models_\partial \phi$ depends on n, so that we might use a more accurate notation $\Theta \models_\partial \phi$. Nevertheless, the following proposition shows that this extra notation is not necessary.

Proposition 3.2.11. *Let $\Theta \subseteq FORM_n$ and $\phi \in FORM_n$. Then for any $m \geq n$*

$$\Theta \models_{n,\partial} \phi \text{ if and only if } \Theta \models_{m,\partial} \phi$$

Proof. One implication is trivial. Conversely, if we suppose that $\Theta \models_{m,\partial} \phi$, since ϕ and Θ are built from a finite set of $n \geq 1$ variables, for the truth of ϕ we need only a finite subset of these variables. Hence $\Theta \models_{n,\partial} \phi$. □

Definition 3.2.12. Given a set $\Theta \subseteq FORM_n$, we denote with $\Theta^{\models_\partial}$ the set of all stable consequences of Θ, in symbols

$$\Theta^{\models_\partial} = \{\phi \in FORM_n \mid \Theta \models_\partial \phi\}$$

Theorem 3.2.13. *Let $\Theta \subseteq FORM_n$. Then $L(\Theta)$ is semisimple if and only if $\Theta^{\models} = \Theta^{\models_\partial} = \Theta^{\vdash}$. Thus $L(\Theta)$ is not semisimple if and only if there is $\phi \in FORM_n$ such that every differential valuation of order 0 satisfying Θ satisfies ϕ and there is a differential valuation $\mathbf{u}$ satisfying Θ but not ϕ.*

Proof. By corollary 3.1.11 together with theorem 3.2.9. □

Theorem 3.2.14. *Let $\Theta \subseteq FORM_n$. Then $L(\Theta)$ is strongly semisimple if and only if for all $\phi \in FORM_n$*

$$(\Theta \cup \{\phi\})^{\models} = (\Theta \cup \{\phi\})^{\models_\partial}$$

Proof. For any Θ' such that $\Theta \subseteq \Theta' \subseteq \Theta^{\vdash}$, by proposition 3.1.7 and proposition 1.6.8, we have $L(\Theta) = L(\Theta') = L(\Theta^{\vdash})$. Whence, without loss of generality, we can assume $\Theta = \Theta^{\vdash}$. Therefore the set $\{1 - f_\theta \mid \theta \in \Theta\}$ is the ideal I_Θ of M_n. Since the map

$$\iota \colon \frac{\phi}{\equiv_\Theta} \in L(\Theta) \to \frac{1 - f_\phi}{I_\Theta} \in \frac{M_n}{I_\Theta}$$

is an isomorphism, the principal ideal $\langle \phi/{\equiv_\Theta} \rangle$ of $L(\Theta)$ corresponds via ι to the principal ideal $\langle \{1 - f_\phi\}/I_\Theta \rangle$ of M_n/I_Θ. Then we have the identities

$$\langle \frac{1 - f_\phi}{I_\Theta} \rangle = \frac{\langle 1 - f_\phi \rangle}{I_\Theta} = \frac{\langle I_\Theta \cup \{1 - f_\phi\} \rangle}{I_\Theta}$$

Therefore $L(\Theta)$ is strongly semisimple iff M_n/I_Θ is strongly semisimple iff for any principal ideal $\langle I_\Theta \cup \{1 - f_\phi\}\rangle/I_\Theta$ of M_n, the quotient

$$\frac{M_n/I_\Theta}{\langle I_\Theta \cup \{1 - f_\phi\}\rangle/I_\Theta} \cong \frac{M_n}{\langle I_\Theta \cup \{1 - f_\phi\}\rangle}$$

is semisimple. It is equivalent to say that $L(\Theta \cup \{\phi\})$ is semisimple for every $\phi \in FORM_n$. Therefore, by theorem 3.2.9, $L(\Theta)$ is strongly semisimple iff $(\Theta \cup \{\phi\})^{\models} = (\Theta \cup \{\phi\})^{\models_\partial}$. $\square$

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